RELATIONS BETWEEN SEMIDUALIZING COMPLEXES

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Abstract. We study the following question: Given two semidualizing complexes B and C over a commutative noetherian ring R, does the vanishing of $\operatorname{Ext}^n_R(B,C)$ for $n \gg 0$ imply that B is C-reflexive? This question is a natural generalization of one studied by Avramov, Buchweitz, and Şega. We begin by providing conditions equivalent to B being C-reflexive, each of which is slightly stronger than the condition $\operatorname{Ext}^n_R(B,C) = 0$ for all $n \gg 0$. We introduce and investigate an equivalence relation $\approx$ on the set of isomorphism classes of semidualizing complexes. This relation is defined in terms of a natural action of the derived Picard group and is well-suited for the study of semidualizing complexes over nonlocal rings. We identify numerous alternate characterizations of this relation, each of which includes the condition $\operatorname{Ext}^n_R(B,C) = 0$ for all $n \gg 0$. Finally, we answer our original question in some special cases.

1. Introduction

Given a dualizing complex D for a commutative noetherian ring R, cohomological properties of D often translate to ring-theoretic properties of R. For example, when R is local, if $\operatorname{Ext}^n_R(D,R) = 0$ for $n \gg 0$ and the natural evaluation morphism $D \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(D,R) \to R$ is an isomorphism in the derived category $\mathcal{D}(R)$, then R is Gorenstein. Recently, Avramov, Buchweitz, and Şega [2] investigated the following potential extensions of this fact.

1.1. Let R be a local ring admitting a dualizing complex D such that $\inf(D) = 0$.

Question. If $\operatorname{Ext}^n_R(D,R) = 0$ for $(\dim(R)+1)$ consecutive values of $n \geqslant 1$, must R be Gorenstein?

Conjecture. If $\operatorname{Ext}^n_R(D,R) = 0$ for all $n \geqslant 1$, then R is Gorenstein.

This paper is concerned with a version of (1.1) for *semi*dualizing complexes[1].

Semidualizing complexes were introduced by Avramov and Foxby [5] in a special case for use in studying local ring homomorphisms, and by Christensen [8] in general. For example, a dualizing complex is semidualizing, as is a free module of rank 1. Each semidualizing complex C yields a duality theory or, more specifically, a notion of C-reflexivity with properties similar to those for reflexivity with respect to D or R; see Section 2 for background material.

Our version of (1.1) for this setting is contained in the following list of questions. Specifically, an affirmative answer to Question 1.2(a) would yield an affirmative

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[1]While this paper is written in the language of complexes and derived categories, our results specialize readily to the case of semidualizing *modules*. For a discussion of the translation from complexes to modules, see [8, (4.10)] and [14, (2.1)].

answer to the question in (1.1), and an affirmative answer to Question 1.2(b) would establish the conjecture in (1.1); see Remark 3.1. Also, note that Example 7.1 shows the need for the local hypothesis in Question 1.2(a).

Question 1.2. Let B and C be semidualizing R-complexes.

(a) If $\mathrm{Ext}^n_R(B,C)=0$ for $(\dim(R)+1)$ consecutive values of $n>\sup(C)-\inf(B)$ and if R is local, must B be C-reflexive?
(b) If $\mathrm{Ext}^n_R(B,C)=0$ for all $n>\sup(C)-\inf(B)$, must B be C-reflexive?
(c) If $\mathrm{Ext}^n_R(B,C)=0$ for $n\gg 0$, must B be C-reflexive?

Our results come in three types. The results of Section 3 are of the first type: Assuming $\mathrm{Ext}^n_R(B,C)=0$ for $n\gg 0$ *and a bit more*, we show that B is C-reflexive. The following is one such result; its proof is in 3.6. Note that each of the conditions (ii)–(iv) includes the condition $\mathrm{Ext}^n_R(B,C)=0$ for $n\gg 0$.

Theorem 1.3. *Let B and C be semidualizing R-complexes. The following conditions are equivalent:*

(i) *B is C-reflexive;*
(ii) *$\mathbf{R}\mathrm{Hom}_R(B,C)$ is semidualizing;*
(iii) *$\mathbf{R}\mathrm{Hom}_R(B,C)$ is C-reflexive;*
(iv) *C is in the Bass class $\mathcal{B}_B(R)$.*

We also prove analogous results for tensor products motivated by the corresponding version of Question 1.2 found in Question 3.9.

Before discussing our second type of results, we describe a new equivalence relation on the set of isomorphism classes of semidualizing R-complexes: We write $[B]\approx[C]$ if there is a tilting R-complex P such that $B\simeq P\otimes^{\mathbf{L}}_R C$. Here, a *tilting R-complex* is a semidualizing R-complex of finite projective dimension. When R is local, the only tilting R-complexes are those of the form $\Sigma^n R$, so in this case $[B]\approx[C]$ if and only if B and C are isomorphic up to shift in $\mathcal{D}(R)$. Hence, our new relation recovers the more standard relation over a local local ring while also being particularly well-suited for the nonlocal setting. Section 4 contains our treatment of tilting complexes and the basics of this relation.

Section 5 contains our results of the second type: Assuming $\mathrm{Ext}^n_R(B,C)=0$ for $n\gg 0$ *and a lot more*, we show that $[B]\approx[C]$. For instance, we prove the next result in 5.2. As with Theorem 1.3, observe that each of the conditions (ii)–(vi) includes the condition $\mathrm{Ext}^n_R(B,C)=0$ for $n\gg 0$.

Theorem 1.4. *Let B and C be semidualizing R-complexes. The following conditions are equivalent:*

(i) *$[B]\approx[C]$;*
(ii) *$\mathbf{R}\mathrm{Hom}_R(B,C)$ is a tilting R-complex;*
(iii) *$\mathbf{R}\mathrm{Hom}_R(B,C)$ has finite projective dimension;*
(iv) *$\mathbf{R}\mathrm{Hom}_R(B,C)$ has finite complete intersection dimension;*
(v) *There is an equality of Bass classes $\mathcal{B}_B(R)=\mathcal{B}_C(R)$;*
(vi) *$B\in\mathcal{B}_C(R)$ and $C\in\mathcal{B}_B(R)$.*

The last two sections contain our results of the third type: analogues of results of Avramov, Buchweitz, and Şega [2]. In Section 6 we consider the question of when the vanishing assumptions in Question 1.2 guarantee that R is Cohen-Macaulay. This lays some of the foundation for the results of Section 7 where we verify special

cases of Question 1.2(a). The primary result of that section is the following theorem whose proof is in 7.2. Recall that R is *generically Gorenstein* if, for each $\mathfrak{p} \in \operatorname{Ass}(R)$, the ring $R_\mathfrak{p}$ is Gorenstein.

Theorem 1.5. *Let R be a local ring that is generically Gorenstein and admits a dualizing complex D such that $\inf(D) = 0$. Fix semidualizing R-complexes B and C such that $\inf(B) = 0 = \inf(C)$. Assume that B is Cohen-Macaulay and $\sup(C) = 0$.*

(a) *If $\operatorname{Ext}^n_R(B, \mathbf{R}\operatorname{Hom}_R(B, D)) = 0$ for $n = 1, \ldots, \dim(R)$, then $B \simeq R$.*
(b) *If $\operatorname{Ext}^n_R(\mathbf{R}\operatorname{Hom}_R(C, D), C) = 0$ for $n = 1, \ldots, \dim(R)$, then $C \simeq D$.*

Note that the special case $B = D$ in part (a) (or $C = R$ in part (b)) is exactly [2, (2.1)]. Of course, we do not extend all of the special cases covered in [2] to the semidualizing arena. Rather, we prove a few results of this type in order to illustrate the natural parallels between the two contexts.

Unlike much of the existing literature on the subject, most of this paper is devoted to the study of semidualizing complexes over nonlocal rings. In a sense, this makes it a natural companion to [14]. However, it should be noted that many of our results are new even in the local case.

2. Complexes

Throughout this paper R is a commutative noetherian ring.

Definition 2.1. We index R-complexes homologically

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

and the *infimum*, *supremum*, and *amplitude* of an R-complex X are

$$\inf(X) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\} \qquad \sup(X) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$$
$$\operatorname{amp}(X) = \sup(X) - \inf(X).$$

The complex X is *homologically bounded* if $\operatorname{amp}(X) < \infty$; it is *degreewise homologically finite* if each R-module $\mathrm{H}_n(X)$ is finitely generated; and it is *homologically finite* if the R-module $\oplus_{n\in\mathbb{Z}} \mathrm{H}_n(X)$ is finitely generated.

For each integer i, the ith *suspension* (or *shift*) of a complex X, denoted $\mathsf{\Sigma}^i X$, is the complex with $(\mathsf{\Sigma}^i X)_n := X_{n-i}$ and $\partial^{\mathsf{\Sigma}^i X}_n := (-1)^i \partial^X_{n-i}$. The notation $\mathsf{\Sigma} X$ is short for $\mathsf{\Sigma}^1 X$. The *projective dimension*, *flat dimension* and *injective dimension* of X are denoted $\operatorname{pd}_R(X)$, $\operatorname{fd}_R(X)$ and $\operatorname{id}_R(X)$, respectively; see [3].

Definition 2.2. We work in the derived category $\mathcal{D}(R)$. References on the subject include [16, 19, 27, 28]. The category $\mathcal{D}_\mathrm{b}(R)$ is the full subcategory of $\mathcal{D}(R)$ consisting of homologically bounded R-complexes. Given two R-complexes X and Y, the derived homomorphism and tensor product complexes are denoted $\mathbf{R}\operatorname{Hom}_R(X, Y)$ and $X \otimes^{\mathbf{L}}_R Y$, respectively. For each integer n, set

$$\operatorname{Ext}^n_R(X, Y) := \mathrm{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(X, Y) \qquad \text{and} \qquad \operatorname{Tor}^R_n(X, Y) := \mathrm{H}_n(X \otimes^{\mathbf{L}}_R Y).$$

Isomorphisms in $\mathcal{D}(R)$ are identified by the symbol $\simeq$, and isomorphisms up to shift are identified by $\sim$.

The *support* and *dimension* of X are, respectively,

$$\operatorname{Supp}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid X_\mathfrak{p} \not\simeq 0\} = \cup_n \operatorname{Supp}_R(\mathrm{H}_n(X))$$
$$\dim_R(X) = \sup\{\dim(R/\mathfrak{p}) - \inf(X_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(X)\}.$$

Definition 2.3. Assume that $(R, \mathfrak{m}, k)$ is a local ring, and let X be an R-complex. The *depth* and *Cohen-Macaulay defect* of X are, respectively

$$\operatorname{depth}_R(X) = -\sup(\mathbf{R}\mathrm{Hom}_R(k, X))$$
$$\operatorname{cmd}_R(X) = \dim_R(X) - \operatorname{depth}_R(X).$$

When X is homologically finite, we have $\operatorname{cmd}_R(X) \geqslant 0$ by [11, (2.8), (3.9)], and X is *Cohen-Macaulay* if $\operatorname{cmd}_R(X) = 0$.

Fact 2.4. For R-complexes X and Y, the following are from [10, (2.1)]

$$\sup(\mathbf{R}\mathrm{Hom}_R(X, Y)) \leqslant \sup(Y) - \inf(X)$$
$$\inf(X \otimes^{\mathbf{L}}_R Y) \geqslant \inf(X) + \inf(Y)$$
$$\operatorname{Ext}_R^{\inf(X)-\sup(Y)}(X, Y) \cong \operatorname{Hom}_R(\mathrm{H}_{\inf(X)}(X), \mathrm{H}_{\sup(Y)}(Y))$$
$$\operatorname{Tor}^R_{\inf(X)+\inf(Y)}(X, Y) \cong \mathrm{H}_{\inf(X)}(X) \otimes_R \mathrm{H}_{\inf(Y)}(Y).$$

Assume that R is local and that X and Y are degreewise homologically finite such that $\inf(X), \inf(Y) > -\infty$. Nakayama's Lemma and the previous display imply $\inf(X \otimes^{\mathbf{L}}_R Y) = \inf(X) + \inf(Y)$.

Fact 2.5. Assume that R is local and that X, Y and Z are degreewise homologically finite R-complexes such that $\inf(X), \inf(Y) > -\infty$ and $\sup(Z) < \infty$. Using [5, (1.5.3)] we see that $\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R Y) < \infty$ if and only if $\operatorname{pd}_R(X) < \infty$ and $\operatorname{pd}_R(Y) < \infty$, and $\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(X, Z)) < \infty$ if and only if $\operatorname{pd}_R(X) < \infty$ and $\operatorname{id}_R(Z) < \infty$.

We shall have several occasions to use the following isomorphisms from [3, (4.4)].

Definition/Notation 2.6. Let X, Y and Z be R-complexes. Assume that X is degreewise homologically finite and $\inf(X) > -\infty$.

The natural *tensor-evaluation morphism*

$$\omega_{XYZ} \colon \mathbf{R}\mathrm{Hom}_R(X, Y) \otimes^{\mathbf{L}}_R Z \to \mathbf{R}\mathrm{Hom}_R(X, Y \otimes^{\mathbf{L}}_R Z)$$

is an isomorphism when $\sup(Y) < \infty$ and either $\operatorname{pd}_R(X) < \infty$ or $\operatorname{pd}_R(Z) < \infty$.

The natural *Hom-evaluation morphism*

$$\theta_{XYZ} \colon X \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Y, Z) \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, Y), Z)$$

is an isomorphism when $Y \in \mathcal{D}_{\mathrm{b}}(R)$ and either $\operatorname{pd}_R(X) < \infty$ or $\operatorname{id}_R(Z) < \infty$.

Semidualizing complexes, defined next, are our main objects of study.

Definition 2.7. A homologically finite R-complex C is *semidualizing* if the natural homothety morphism $\chi^R_C \colon R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathcal{D}(R)$. An R-complex D is *dualizing* if it is semidualizing and $\operatorname{id}_R(D) < \infty$. Let $\overline{\mathfrak{S}}(R)$ denote the set of isomorphism classes of semidualizing R-complexes.

Remark 2.8. In some of the literature, the set of *shift*-isomorphism classes of semidualizing R-complexes is denoted $\mathfrak{S}(R)$. The notation $\overline{\mathfrak{S}}(R)$ is meant to evoke the notation $\mathfrak{S}(R)$ while at the same time distinguishing between the two notations.

We include the following properties for ease of reference.

Properties 2.9. Let C be a semidualizing R-complex.

2.9.1. The R-module R is R-semidualizing. When R is local, we have $\operatorname{pd}_R(C) < \infty$ if and only if $C \sim R$ by [8, (8.1)].

2.9.2. If R is Gorenstein and local, then $C \sim R$. Conversely, if R is dualizing for R, then R is Gorenstein. See [8, (8.6)] and [19, (V.9)].

2.9.3. If X is a homologically finite R-complex, then X is semidualizing for R if and only if $X_{\mathfrak{m}}$ is semidualizing for $R_{\mathfrak{m}}$ for each maximal (equivalently, for each prime) ideal $\mathfrak{m} \subset R$; see [14, (2.3)]. When R is local and C is semidualizing with $s = \sup(C)$, if $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{H}_s(C))$, then $\inf(C_{\mathfrak{p}}) = s$ by [8, (A.7)].

2.9.4. Let $\varphi\colon R \to S$ be a local homomorphism of finite flat dimension, and fix semidualizing R-complexes B, C. The complex $S \otimes^{\mathbf{L}}_R C$ is semidualizing for S, and $\operatorname{amp}(S \otimes^{\mathbf{L}}_R C) = \operatorname{amp}(C)$ by [8, (5.7)]. If $S \otimes^{\mathbf{L}}_R C$ is dualizing for S, then C is dualizing for R by [4, (4.2), (5.1)]. Conversely, if C is dualizing for R and φ is surjective with kernel generated by an R-sequence, then $S \otimes^{\mathbf{L}}_R C$ is dualizing for S, and $\widehat{R} \otimes^{\mathbf{L}}_R C$ is dualizing for $\widehat{R}$ by [4, (4.2), (4.3), (5.1)]. Finally, if $S \otimes^{\mathbf{L}}_R B \simeq S \otimes^{\mathbf{L}}_R C$ in $\mathcal{D}(S)$, then [14, (1.10)] implies that $B \simeq C$ in $\mathcal{D}(R)$.

2.9.5. Let $\alpha\colon X \to Y$ be a morphism between degreewise homologically finite R-complexes. Assuming $\inf(X), \inf(Y) > -\infty$, if $\mathbf{R}\mathrm{Hom}(\alpha, C)$ is an isomorphism in $\mathcal{D}(R)$, then so is α. Dually, assuming $\sup(X), \sup(Y) < \infty$, if $\mathbf{R}\mathrm{Hom}(C, \alpha)$ is an isomorphism in $\mathcal{D}(R)$, then so is α. These follow from [5, (1.2.3.b)] and [7, (A.8.11), (A.8.13)] as the isomorphism $R \simeq \mathbf{R}\mathrm{Hom}_R(C, C)$ implies $\operatorname{Supp}_R(C) = \operatorname{Spec}(R)$.

2.9.6. If C is a module, then an element of R is C-regular if and only if it is R-regular as the isomorphism $R \cong \operatorname{Hom}_R(C, C)$ implies $\operatorname{Ass}_R(C) = \operatorname{Ass}(R)$.

2.9.7. When R is local, there are inequalities

$$\max\{\operatorname{amp}(C), \operatorname{cmd}_R(C)\} \leqslant \operatorname{cmd}(R) \leqslant \operatorname{amp}(C) + \operatorname{cmd}_R(C)$$

by [8, (3.4)]. In particular, if R is Cohen-Macaulay and local, then $\operatorname{amp}(C) = 0$.

The next definition is from [8] and [19] and will be used primarily to compare semidualizing complexes.

Definition 2.10. Let C be a semidualizing R-complex. A homologically finite R-complex X is *C-reflexive* when it satisfies the following:

(1) $\operatorname{Ext}^n_R(X, C) = 0$ for $n \gg 0$, and
(2) the natural biduality morphism $\delta^C_X\colon X \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, C), C)$ is an isomorphism in $\mathcal{D}(R)$.

The following properties are frequently used in the sequel.

Properties 2.11. Let B and C be semidualizing R-complexes.

2.11.1. Each homologically finite R-complex of finite projective dimension is C-reflexive by [14, (3.11)].

2.11.2. Let D be a dualizing R-complex, and X a homologically finite R-complex. The complex X is D-reflexive by [19, (V.2.1)], and X is semidualizing if and only if $\mathbf{R}\mathrm{Hom}_R(X, D)$ is semidualizing by [8, (2.12)] and (2.9.3). The natural evaluation morphism $C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, D) \to D$ is an isomorphism in $\mathcal{D}(R)$ by [14, (3.1.b)].

Assume that R is local and $\inf(C) = 0 = \inf(D)$. With the isomorphism from the previous paragraph, Fact 2.4 yields the second equality in the next sequence

$$0 = \inf(D) = \inf(C) + \inf(\mathbf{R}\mathrm{Hom}_R(C, D)) = \inf(\mathbf{R}\mathrm{Hom}_R(C, D))$$

and [11, (3.14)] implies $\operatorname{amp}(\mathbf{R}\mathrm{Hom}_R(X, D)) = \operatorname{cmd}(X)$.

2.11.3. If X is a homologically finite R-complex, then X is C-reflexive if and only if $\operatorname{Ext}^i_R(X,C)=0$ for $i \gg 0$ and $X_{\mathfrak{m}}$ is $C_{\mathfrak{m}}$-reflexive for each maximal (equivalently, for each prime) ideal $\mathfrak{m} \subset R$; see [14, (2.8)].

2.11.4. If R is local, then B is C-reflexive and C is B-reflexive if and only if $B \sim C$ by [1, (5.5)].

2.11.5. When $B \otimes^{\mathbf{L}}_R C$ is also semidualizing we know from [14, (3.1.c)] that C is $B \otimes^{\mathbf{L}}_R C$-reflexive and furthermore $\mathbf{R}\mathrm{Hom}_R(C, B \otimes^{\mathbf{L}}_R C) \simeq B$.

The following categories, known collectively as "Foxby classes", were introduced by Foxby [13], Avramov and Foxby [5], and Christensen [8].

Definition 2.12. Let C be a semidualizing R-complex. The *Auslander class* associated to C, denoted $\mathcal{A}_C(R)$, is the full subcategory of $\mathcal{D}(R)$ consisting of all complexes X satisfying the following conditions:

(1) X and $C \otimes^{\mathbf{L}}_R X$ are homologically bounded, and
(2) the natural morphism $\gamma^C_X \colon X \to \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$ is an isomorphism.

The *Bass class* associated to C, denoted $\mathcal{B}_C(R)$, is the full subcategory of $\mathcal{D}(R)$ consisting of all complexes X satisfying the following conditions:

(1) X and $\mathbf{R}\mathrm{Hom}_R(C,X)$ are homologically bounded, and
(2) the evaluation morphism $\xi^C_X \colon C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X) \to X$ is an isomorphism.

Fact 2.13. Let C be a semidualizing R-complex. From [8, (4.6)] we conclude that a homologically bounded complex X is in $\mathcal{A}_C(R)$ if and only if $C \otimes^{\mathbf{L}}_R X$ is in $\mathcal{B}_C(R)$, and dually, a homologically bounded complex X is in $\mathcal{B}_C(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(C,X)$ is in $\mathcal{A}_C(R)$. Each R-complex of finite injective dimension is in $\mathcal{B}_C(R)$, and each R-complex of finite projective dimension is in $\mathcal{A}_C(R)$ by [8, (4.4)].

The complete intersection dimensions used in this paper were defined for modules by Avramov, Gasharov and Peeva [6] and Sahandi, Sharif and Yassemi [24], and then for complexes by Sather-Wagstaff [25, 26]. We start with the definitions, first over a local ring and then in general.

Definition 2.14. Assume that R is local. A *quasi-deformation* of R is a diagram of local ring homomorphisms $R \xrightarrow{\varphi} R' \xleftarrow{\rho} Q$ in which φ is flat and ρ is surjective with $\operatorname{Ker}(\rho)$ generated by a Q-regular sequence.

For each homologically finite R-complex X, define the *complete intersection dimension* and *complete intersection injective dimension* of X as follows.

$$\text{CI-dim}_R(X) := \inf\left\{ \mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_R X) - \mathrm{pd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{quasi-deformation} \end{array} \right\}$$
$$\text{CI-id}_R(X) := \inf\left\{ \mathrm{id}_Q(R' \otimes^{\mathbf{L}}_R X) - \mathrm{pd}_Q(R') \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is a} \\ \text{quasi-deformation} \end{array} \right\}$$

Definition 2.15. For each homologically finite R-complex X, define the *complete intersection dimension* and *complete intersection injective dimension* of X as

$$\text{CI-dim}_R(X) := \sup\{\text{CI-dim}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) \mid \mathfrak{m} \subset R \text{ is a maximal ideal}\}$$
$$\text{CI-id}_R(X) := \sup\{\text{CI-id}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) \mid \mathfrak{m} \subset R \text{ is a maximal ideal}\}.$$

Fact 2.16. Let X be a homologically finite R-complex. If $\mathrm{pd}_R(X) < \infty$, then $\text{CI-dim}_R(X) < \infty$ by [26, (3.3)]. If R is local and $\mathrm{id}_R(X) < \infty$, then the trivial

quasi-deformation $R \to R \leftarrow R$ yields $\text{CI-id}_R(X) < \infty$; see Lemma A.1 for the nonlocal situation.

Assume that R is local. One checks readily that $\text{CI-dim}_R(X)$ is finite if and only if $\text{CI-dim}_{\widehat{R}}(\widehat{R} \otimes^{\mathbf{L}}_R X)$ is finite. If X has finite length homology, then $\text{CI-id}_R(X)$ is finite if and only if $\text{CI-id}_{\widehat{R}}(\widehat{R} \otimes^{\mathbf{L}}_R X)$ is finite by [25, (3.7)].

We document several useful but independent facts about quasi-deformations and complete intersection dimensions in Appendix A.

3. Detecting Reflexivity

The results of this section are of the first type discussed in the introduction: Assuming $\text{Ext}^n_R(B, C) = 0$ for $n \gg 0$ *and a bit more*, we show that B is C-reflexive. We begin, though, by discussing the link between (1.1) and Question 1.2.

Remark 3.1. Assume that the answer to Question 1.2(a) is "yes". Then the answer to the question in (1.1) is also "yes". Indeed, let D be a dualizing R-complex such that $\inf(D) = 0$, and assume that $\text{Ext}^n_R(D, R) = 0$ for $(\dim(R) + 1)$ consecutive values of $n \geqslant 1$. The R-complexes R and D are semidualizing, and $\sup(R) = 0 = \inf(D)$. Thus, the affirmative answer to Question 1.2(a) implies that D is R-reflexive. From (2.11.2) we know that R is D-reflexive, and so (2.11.4) implies $R \sim D$. Using (2.9.2) we conclude that R is Gorenstein, as desired.

Similarly, if the answer to Question 1.2(b) is "yes", then this would establish the conjecture in (1.1).

Before proceeding, we briefly discuss the necessity of the semidualizing hypothesis in Question 1.2.

Remark 3.2. If the semidualizing-hypothesis is removed from Question 1.2, then the answer to each of the resulting questions is "no". Specifically, Jorgensen and Şega [23, Theorem] exhibit a local ring R and a family $\{M_s\}_{2 \leqslant s \leqslant \infty}$ of R-modules such that $\text{Ext}^n_R(M_s, R) = 0$ for $i = 1, \ldots, s$ and M_s is not R-reflexive, for each s.

When R is local, the forward implication in the following result is in [8, (2.11)], but the proof of [8, (2.11)] makes no use of the local hypothesis. For the reverse implication, argue as in [7, (2.1.10)].

Lemma 3.3. *Let X and C be homologically finite R-complexes with C semidualizing. The complex X is C-reflexive if and only if $\mathbf{R}\text{Hom}_R(X, C)$ is C-reflexive.* $\square$

The following diagram will be used in the next two proofs.

Remark 3.4. Let X and C be homologically finite R-complexes with C semidualizing. There is a commutative diagram of morphisms of complexes

$$\begin{tikzcd} R \arrow[r, "\chi^R_{\mathbf{R}\text{Hom}_R(X,C)}"] \arrow[d, "\chi^R_X"'] & \mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(X,C), \mathbf{R}\text{Hom}_R(X,C)) \arrow[d, "\simeq"] \\ \mathbf{R}\text{Hom}_R(X,X) \arrow[r, "\mathbf{R}\text{Hom}_R(X,\delta^C_X)"] & \mathbf{R}\text{Hom}_R(X, \mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(X,C), C)) \end{tikzcd}$$

wherein the unspecified arrow is the "swap" isomorphism from [7, (A.2.9)].

Before proving Theorem 1.3, we present a similar result, where only one of the complexes is assumed to be semidualizing. When R is local, the implication (i) $\Longrightarrow$ (ii) is in [8, (2.11)], but the proof of [8, (2.11)] makes no use of the local hypothesis. Also, the symmetry of the conditions in our result suggest a fourth condition, namely, "X is semidualizing and $\mathbf{R}\mathrm{Hom}_R(X,C)$ is C-reflexive"; this is shown to be equivalent in Theorem 1.3.

Proposition 3.5. *Let X and C be homologically finite R-complexes with C semidualizing. The following conditions are equivalent:*

(i) *X is semidualizing and C-reflexive;*
(ii) *$\mathbf{R}\mathrm{Hom}_R(X,C)$ is semidualizing and C-reflexive;*
(iii) *$\mathbf{R}\mathrm{Hom}_R(X,C)$ is semidualizing and X is C-reflexive.*

Proof. As noted above, the implication (i) $\Longrightarrow$ (ii) is proved as in [8, (2.11)]. Also, the equivalence (ii) $\Longleftrightarrow$ (iii) is from Lemma 3.3.

(iii) $\Longrightarrow$ (i). Assume that $\mathbf{R}\mathrm{Hom}_R(X,C)$ is semidualizing and X is C-reflexive. Then the morphisms $\chi^R_{\mathbf{R}\mathrm{Hom}_R(X,C)}$ and δ^C_X are isomorphisms. Hence, the morphism $\mathbf{R}\mathrm{Hom}_R(X,\delta^C_X)$ is an isomorphism, and so the diagram in Remark 3.4 shows that χ^R_X is also an isomorphism. By definition, we conclude that X is semidualizing. $\square$

3.6. *Proof of Theorem 1.3.* (i) $\Longleftrightarrow$ (iii). This is from Lemma 3.3.

(i) $\Longleftrightarrow$ (ii). The conditions (i) and (ii) each imply that $\mathbf{R}\mathrm{Hom}_R(B,C)$ is homologically bounded. So, it remains to assume that $\mathbf{R}\mathrm{Hom}_R(B,C)$ is homologically bounded and show that $\chi^R_{\mathbf{R}\mathrm{Hom}_R(B,C)}$ and δ^C_B are isomorphisms simultaneously.

As B is semidualizing, the morphism χ^R_B is an isomorphism. Hence the diagram from Remark 3.4 with $X = B$ shows that $\chi^R_{\mathbf{R}\mathrm{Hom}_R(B,C)}$ and $\mathbf{R}\mathrm{Hom}_R(B,\delta^C_B)$ are isomorphisms simultaneously. From (2.9.5), we know that $\mathbf{R}\mathrm{Hom}_R(B,\delta^C_B)$ is an isomorphism if and only if δ^C_B is so, and hence the desired equivalence.

(ii) $\Longleftrightarrow$ (iv). Consider the following commutative diagram of morphisms of complexes wherein the unspecified isomorphism is a combination of Hom-tensor adjointness and commutativity of tensor product.

$$\begin{tikzcd}
R \arrow[r, "\chi^R_{\mathbf{R}\mathrm{Hom}_R(B,C)}"] \arrow[d, "\chi^R_C"', "\simeq"] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(B,C),\mathbf{R}\mathrm{Hom}_R(B,C)) \arrow[d, "\simeq"] \\
\mathbf{R}\mathrm{Hom}_R(C,C) \arrow[r, "\mathbf{R}\mathrm{Hom}_R(\xi^B_C,C)"] & \mathbf{R}\mathrm{Hom}_R(B\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B,C),C)
\end{tikzcd}$$

Using this diagram with (2.9.5), the desired equivalence is established as in the proof of (i) $\Longleftrightarrow$ (ii). $\square$

Next we prove versions of Proposition 3.5 and Theorem 1.3 for Auslander classes.

Proposition 3.7. *Let X and C be homologically finite R-complexes with C semidualizing. The following conditions are equivalent:*

(i) *X is semidualizing and in $\mathcal{A}_C(R)$;*
(ii) *X and $C\otimes^{\mathbf{L}}_R X$ are semidualizing;*
(iii) *X is in $\mathcal{A}_C(R)$ and $C\otimes^{\mathbf{L}}_R X$ is semidualizing;*
(iv) *$C\otimes^{\mathbf{L}}_R X$ is semidualizing and C is $C\otimes^{\mathbf{L}}_R X$-reflexive;*
(v) *$C\otimes^{\mathbf{L}}_R X$ is semidualizing and X is $C\otimes^{\mathbf{L}}_R X$-reflexive.*

Proof. Consider the following commutative diagram of morphisms of complexes

$$\begin{tikzcd} R \arrow[r, "\chi^R_{C\otimes^{\mathbf{L}}_R X}"] \arrow[d, "\chi^R_X"'] & \mathbf{R}\mathrm{Hom}_R(C\otimes^{\mathbf{L}}_R X, C\otimes^{\mathbf{L}}_R X) \arrow[d, "\simeq"] \\ \mathbf{R}\mathrm{Hom}_R(X,X) \arrow[r, "\mathbf{R}\mathrm{Hom}_R(X,\gamma^C_X)"] & \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_R(C, C\otimes^{\mathbf{L}}_R X)) \end{tikzcd}$$

wherein the unspecified isomorphism is a combination of Hom-tensor adjointness and commutativity of tensor product.

(i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii). Use the above diagram as in (3.6).

(iii) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (v). Assume that X is in $\mathcal{A}_C(R)$ and $C\otimes^{\mathbf{L}}_R X$ is semidualizing. Using the above diagram, we see that X is semidualizing, and (2.11.5) implies that C and X are $C\otimes^{\mathbf{L}}_R X$-reflexive.

(iv) $\Longrightarrow$ (iii). Assume that $C\otimes^{\mathbf{L}}_R X$ is semidualizing and C is $C\otimes^{\mathbf{L}}_R X$-reflexive. Theorem 1.3 implies $C\otimes^{\mathbf{L}}_R X\in\mathcal{B}_C(R)$, and so Fact 2.13 implies $X\in\mathcal{A}_C(R)$.

(v) $\Longrightarrow$ (iv). Assume that $C\otimes^{\mathbf{L}}_R X$ is semidualizing and X is $C\otimes^{\mathbf{L}}_R X$-reflexive. The morphism $\gamma^X_C\colon C\to\mathbf{R}\mathrm{Hom}_R(X, X\otimes^{\mathbf{L}}_R C)$ is locally an isomorphism by a result of Gerko [18, (3.5)], and hence γ^X_C is an isomorphism. Since X is $C\otimes^{\mathbf{L}}_R X$-reflexive, we conclude from Lemma 3.3 that the following complex is also $C\otimes^{\mathbf{L}}_R X$-reflexive

$$\mathbf{R}\mathrm{Hom}_R(X, C\otimes^{\mathbf{L}}_R X)\simeq\mathbf{R}\mathrm{Hom}_R(X, X\otimes^{\mathbf{L}}_R C)\simeq C. \qquad \square$$

Corollary 3.8. *Let B and C be semidualizing R-complexes. The following conditions are equivalent:*

(i) *$C\otimes^{\mathbf{L}}_R B$ is semidualizing;*
(ii) *B is in $\mathcal{A}_C(R)$;*
(iii) *C is in $\mathcal{A}_B(R)$.* $\square$

Observe that the implication (ii) $\Longrightarrow$ (i) in Corollary 3.8 has the following form: Assuming $\mathrm{Tor}^R_n(B,C)=0$ for $n\gg 0$ *and a bit more*, we conclude that $B\otimes^{\mathbf{L}}_R C$ is semidualizing. In light of Question 1.2 and Theorem 1.3, this motivates the next question. Regarding the hypotheses of Question 3.9(a), Proposition 6.6 gives a partial justification of the range of Tor-vanishing, and Example 7.1 shows why R must be local.

Question 3.9. Let B and C be semidualizing R-complexes.

(a) If $\mathrm{Tor}^R_n(B,C)=0$ for $(2\dim(R)+1)$ consecutive values of $n>\inf(B)+\inf(C)$ and R is local, must $B\otimes^{\mathbf{L}}_R C$ be semidualizing?
(b) If $\mathrm{Tor}^R_n(B,C)=0$ for all $n>\inf(B)+\inf(C)$, must $B\otimes^{\mathbf{L}}_R C$ be semidualizing?
(c) If $\mathrm{Tor}^R_n(B,C)=0$ for $n\gg 0$, must $B\otimes^{\mathbf{L}}_R C$ be semidualizing?

We raise the next questions in light of Propositions 3.5 and 3.7.

Question 3.10. Let X and C be homologically finite R-complexes, and assume that C is semidualizing.

(a) If $C\otimes^{\mathbf{L}}_R X$ is semidualizing, must X also be semidualizing?
(b) If $\mathbf{R}\mathrm{Hom}_R(C,X)$ is semidualizing, must X also be semidualizing?
(c) If $\mathbf{R}\mathrm{Hom}_R(X,C)$ is semidualizing, must X also be semidualizing?

Remark 3.11. As with Question 1.2, if we assume more in Question 3.10, then we have an affirmative answer. For instance, if CI-$\dim_R(X)$ is finite and $C\otimes^{\mathbf{L}}_R X$ is

semidualizing, then X is also semidualizing. Indeed, using (2.9.3) we may assume without loss of generality that R is local. The finiteness of $\text{CI-dim}_R(X)$ implies $X \in \mathcal{A}_C(R)$ by [25, (5.1.a)]. Hence, Proposition 3.7 implies that X is semidualizing.

Similarly, if either (1) $\text{CI-dim}_R(X) < \infty$ and $\mathbf{R}\text{Hom}_R(X, C)$ is semidualizing or (2) $\text{CI-id}_R(X) < \infty$ and $\mathbf{R}\text{Hom}_R(C, X)$ is semidualizing, then X is semidualizing.

The final result of this section shows that Questions 1.2(c) and 3.9(c) are equivalent when R admits a dualizing complex.

Proposition 3.12. *Assume that R admits a dualizing complex D. The following conditions are equivalent:*

(i) *For all semidualizing R-complexes B and C, if $\text{Ext}^n_R(B, C) = 0$ for $n \gg 0$, then $\mathbf{R}\text{Hom}_R(B, C)$ is semidualizing;*
(ii) *For all semidualizing R-complexes B and C, if $\text{Tor}^R_n(B, C) = 0$ for $n \gg 0$, then $B \otimes^{\mathbf{L}}_R C$ is semidualizing.*

Proof. (i) $\Longrightarrow$ (ii). Assume $\text{Tor}^R_n(B, C) = 0$ for $n \gg 0$. This means that the complex $B \otimes^{\mathbf{L}}_R C$ is homologically finite. It follows that the same is true for the complexes in the next display where the isomorphism is Hom-tensor adjointness

$$\mathbf{R}\text{Hom}_R(B \otimes^{\mathbf{L}}_R C, D) \simeq \mathbf{R}\text{Hom}_R(B, \mathbf{R}\text{Hom}_R(C, D)).$$

The homological finiteness of the second complex says $\text{Ext}^n_R(B, \mathbf{R}\text{Hom}_R(C, D)) = 0$ for $n \gg 0$. Because the complexes B and $\mathbf{R}\text{Hom}_R(C, D)$ are semidualizing, condition (i) implies that $\mathbf{R}\text{Hom}_R(B, \mathbf{R}\text{Hom}_R(C, D))$ is semidualizing. The displayed isomorphism implies that $\mathbf{R}\text{Hom}_R(B \otimes^{\mathbf{L}}_R C, D)$ is semidualizing, and so $B \otimes^{\mathbf{L}}_R C$ is semidualizing by (2.11.2).

(ii) $\Longrightarrow$ (i). The proof is similar to the previous paragraph using the following Hom-evaluation isomorphism (2.6) in place of adjointness

$$\mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(B, C), D) \simeq B \otimes^{\mathbf{L}}_R \mathbf{R}\text{Hom}_R(C, D). \qquad \square$$

4. Derived Picard Group Action and Associated Relation

The goal of this section is to discuss a relation $\approx$ that is better suited than $\sim$ for studying semidualizing complexes over nonlocal rings. As motivation for our first definition, recall that a *line bundle* on R is a finitely generated locally free (i.e., projective) R-module of rank 1. The *Picard group* of R, denoted $\text{Pic}(R)$, is the set of isomorphism classes of line bundles on R. Given a line bundle P, its class in $\text{Pic}(R)$ is denoted $[P]$. As the name "Picard group" suggests, $\text{Pic}(R)$ carries the structure of an abelian group with additive identity $[R]$ and operations

$$[P] + [Q] = [P \otimes_R Q] \qquad \text{and} \qquad [P] - [Q] = [\text{Hom}_R(Q, P)].$$

When R is local, its Picard group is trivial, that is, we have $\text{Pic}(R) = \{[R]\}$.

Definition 4.1. A *tilting R-complex* is a homologically finite R-complex P of finite projective dimension such that $P_{\mathfrak{p}} \sim R_{\mathfrak{p}}$ for each $\mathfrak{p} \in \text{Spec}(R)$. The *derived Picard group* of R, denoted $\text{DPic}(R)$, is the set of isomorphism classes of tilting R-complexes, and the class of a tilting R-complex P in $\text{DPic}(R)$ is denoted $[P]$. For each $[P], [Q] \in \text{DPic}(R)$, set

$$[P] + [Q] = [P \otimes^{\mathbf{L}}_R Q] \qquad \text{and} \qquad [P] - [Q] = [\mathbf{R}\text{Hom}_R(Q, P)].$$

Remark 4.2. Definition 4.1 is inspired by the derived Picard group of Yekutieli [29]. Note that our definition differs from Yekutieli's in that we do not assume that R contains a field and the definition does not depend on R being an algebra.

The following properties are verified using routine arguments.

Properties 4.3. Let C be a semidualizing R-complex.

4.3.1. The operations defined in (4.1) endow $\mathrm{DPic}(R)$ with the structure of an abelian group with identity $[R]$. Tensor-evaluation (2.6) shows that $[\mathbf{R}\mathrm{Hom}_R(P,R)]$ is an additive inverse for $[P] \in \mathrm{DPic}(R)$.

4.3.2. Each line bundle on R is a tilting complex, and this yields a well-defined injective abelian group homomorphism $\mathrm{Pic}(R) \hookrightarrow \mathrm{DPic}(R)$ given by $[P] \mapsto [P]$.

4.3.3. Each tilting R-complex P is homologically finite and locally semidualizing, so (2.9.3) implies P is semidualizing. This yields a well-defined injective map $\mathrm{DPic}(R) \hookrightarrow \overline{\mathfrak{S}}(R)$ given by $[P] \mapsto [P]$.

4.3.4. If R is local, then $\mathrm{DPic}(R) = \{[\Sigma^n R] \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$.

The first result of this section shows that, in order to verify that an R-complex is tilting, one need not check that it has finite projective dimension.

Proposition 4.4. *A homologically finite R-complex X is tilting if and only if $X_\mathfrak{m} \sim R_\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \subset R$.*

Proof. For the nontrivial implication, assume $X_\mathfrak{m} \sim R_\mathfrak{m}$ for each maximal ideal $\mathfrak{m} \subset R$. It follows that $X_\mathfrak{p} \sim R_\mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$. Because X is homologically finite, this implies

$$\mathrm{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}) = \sup(X_\mathfrak{p}) \leqslant \sup(X).$$

From this we deduce the first inequality in the next sequence

$$\mathrm{pd}_R(X) = \sup_\mathfrak{p}(\mathrm{pd}_{R_\mathfrak{p}}(X_\mathfrak{p})) \leqslant \sup(X) < \infty$$

while the equality is from [3, (5.3.P)]. Thus X is a tilting R-complex. $\square$

Next, we prove a nonlocal version of [8, (8.1)].

Lemma 4.5. *Let C be a semidualizing R-complex. The following are equivalent:*

(i) *C is a tilting R-complex;*
(ii) *$\mathcal{B}_C(R) = \mathcal{D}_\mathrm{b}(R)$;*
(iii) *$R \in \mathcal{B}_C(R)$;*
(iv) *$\mathcal{A}_C(R) = \mathcal{D}_\mathrm{b}(R)$;*
(v) *$E_R(R/\mathfrak{m}) \in \mathcal{A}_C(R)$ for each maximal ideal $\mathfrak{m} \subset R$.*

Proof. (i) $\Longrightarrow$ (ii). Assume that C is a tilting R-complex. The containment $\mathcal{B}_C(R) \subseteq \mathcal{D}_\mathrm{b}(R)$ is by definition. For the reverse containment, fix a complex $X \in \mathcal{D}_\mathrm{b}(R)$. Because C and X are both homologically bounded and C has finite projective dimension we conclude that $\mathbf{R}\mathrm{Hom}_R(C,X)$ is homologically bounded. Consider the commutative diagram

$$\begin{tikzcd}
C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X) \arrow[r, "\theta_{CCX}", "\simeq"'] \arrow[d, "\xi^C_X"'] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C,C),X) \arrow[d, "\mathbf{R}\mathrm{Hom}_R(\chi^R_C,X)", "\simeq"'] \\
X & \mathbf{R}\mathrm{Hom}_R(R,X) \arrow[l, "\zeta_X"', "\simeq"]
\end{tikzcd}$$

wherein θ_{CCX} is the Hom-evaluation isomorphism (2.6), χ^R_C is the homothety isomorphism (4.3.3), and ζ_X is the natural evaluation isomorphism. The diagram shows that ξ^C_X is an isomorphism, and so $X \in \mathcal{B}_C(R)$.

(ii) $\Longrightarrow$ (iii). This is immediate from the condition $R \in \mathcal{D}_{\mathrm{b}}(R)$.

(iii) $\Longrightarrow$ (i). Assume that R is in $\mathcal{B}_C(R)$. Because C is homologically finite, the condition $R \in \mathcal{B}_C(R)$ implies $R_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ for each maximal ideal $\mathfrak{m} \subset R$. Hence, the local version of this result [8, (8.1)] yields $C_\mathfrak{m} \sim R_\mathfrak{m}$ for each $\mathfrak{m}$, and so C is tilting by Proposition 4.4.

The equivalences (i) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) are established similarly. $\square$

The following is for use in the proofs of Theorem 1.4 and Proposition 5.5.

Proposition 4.6. *Let C be a semidualizing R-complex.*

(a) $\text{CI-dim}_R(C) < \infty$ *if and only if C is tilting.*
(b) $\text{CI-id}_R(C) < \infty$ *if and only if C is dualizing.*

Proof. (a) One implication is in Fact 2.16. For the other implication, assume that $\text{CI-dim}_R(C)$ is finite. It follows that $\text{CI-dim}_{R_\mathfrak{m}}(C_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m} \subset R$. If we can show that this implies $C_\mathfrak{m} \sim R_\mathfrak{m}$ for each $\mathfrak{m}$, then Proposition 4.4 will imply that C is tilting. Hence, we may replace R and C by $R_\mathfrak{m}$ and $C_\mathfrak{m}$ in order to assume that R is local.

Because R is local and $\text{CI-dim}_R(C)$ is finite, there exists a quasi-deformation $R \to R' \leftarrow Q$ such that $\mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_R C)$ is finite and Q is complete; see [25, Thm. F]. Because Q is complete, Lemma A.2 implies that there exists a semidualizing Q-complex N such that $R' \otimes^{\mathbf{L}}_R C \simeq R' \otimes^{\mathbf{L}}_Q N$. The finiteness of $\mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_Q N) = \mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_R C)$ implies $\mathrm{pd}_Q(N) < \infty$ by Fact 2.5, and so $N \sim Q$ by (2.9.1). Hence, the isomorphism $C \sim R$ follows from Lemma A.4(a).

(b) For the nontrivial implication, assume $\text{CI-id}_R(C) < \infty$. By Lemma A.1(b), it suffices to show that $\mathrm{id}_{R_\mathfrak{m}}(C_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m} \subset R$. Hence, we may assume, as in the proof of part (a), that R is local with maximal ideal $\mathfrak{m}$.

Fix a minimal generating sequence $\mathbf{x}$ for $\mathfrak{m}$, and consider the Koszul complex $K = K^R(\mathbf{x})$. The finiteness of $\text{CI-id}_R(C)$ and $\mathrm{pd}_R(K)$ implies $\text{CI-id}_R(K \otimes^{\mathbf{L}}_R C) < \infty$ by [25, (4.4.b)]. The complex $K \otimes^{\mathbf{L}}_R C$ has finite length homology, so [25, (3.6)] implies that there exists a quasi-deformation $R \xrightarrow{\varphi} R' \xleftarrow{\rho} Q$ such that Q is complete and $\mathrm{id}_Q(R' \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R C))$ is finite. Again invoking Lemma A.2, there is a semidualizing Q-complex N such that $R' \otimes^{\mathbf{L}}_Q N \simeq R' \otimes^{\mathbf{L}}_R C$. Lemma A.3 provides a sequence $\mathbf{y} \in Q$ such that the Koszul complex $L = K^Q(\mathbf{y})$ satisfies $R' \otimes^{\mathbf{L}}_Q L \simeq R' \otimes^{\mathbf{L}}_R K$. By Lemma A.5 there is an isomorphism

$$R' \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R C) \simeq \mathbf{R}\mathrm{Hom}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q), N)$$

and so the finiteness of

$$\mathrm{id}_Q(R' \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R C)) = \mathrm{id}_Q(\mathbf{R}\mathrm{Hom}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q), N))$$

implies $\mathrm{id}_Q(N) < \infty$ by Fact 2.5. Because N is semidualizing for Q, this means that N is dualizing for Q, and Lemma A.4(b) implies that C is dualizing for R. $\square$

Remark 4.7. Let X be an R-complex. By combining Fact 2.16 with Property 4.3.3 and Proposition 4.6(a), one concludes that the following conditions are equivalent:

(i) X is tilting;
(ii) X is semidualizing and $\mathrm{pd}_R(X)$ is finite;

(iii) X is semidualizing and $\text{CI-dim}_R(X)$ is finite.

The next result sets the stage for our equivalence relation on $\overline{\mathfrak{S}}(R)$.

Proposition 4.8. *Let P be a tilting R-complex, and let B, C, and X be R-complexes with B and C semidualizing.*

(a) *The R-complex X is tilting (respectively, semidualizing or dualizing) if and only if $P \otimes^{\mathbf{L}}_R X$ is so.*
(b) *There are isomorphisms*

$$\mathbf{R}\mathrm{Hom}_R(C, P \otimes^{\mathbf{L}}_R C) \simeq P \qquad \textit{and} \qquad \mathbf{R}\mathrm{Hom}_R(P \otimes^{\mathbf{L}}_R C, C) \simeq \mathbf{R}\mathrm{Hom}_R(P, R).$$

(c) *If $P \simeq \mathbf{R}\mathrm{Hom}_R(B, C)$, then $C \simeq P \otimes^{\mathbf{L}}_R B$.*

Proof. (a) For each maximal ideal $\mathfrak{m} \subset R$, we have

$$(X \otimes^{\mathbf{L}}_R P)_{\mathfrak{m}} \simeq X_{\mathfrak{m}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} P_{\mathfrak{m}} \sim X_{\mathfrak{m}} \otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} R_{\mathfrak{m}} \simeq X_{\mathfrak{m}}.$$

Hence, Proposition 4.4 shows that X is tilting if and only if $P \otimes^{\mathbf{L}}_R X$ is tilting; and X is semidualizing if and only if $P \otimes^{\mathbf{L}}_R X$ is semidualizing by (2.9.3). If X is dualizing, then the previous sentence shows that $P \otimes^{\mathbf{L}}_R X$ is semidualizing, and [3, (4.5.F)] implies $\mathrm{id}_R(P \otimes^{\mathbf{L}}_R X) < \infty$; that is, $P \otimes^{\mathbf{L}}_R X$ is dualizing. Conversely, if $P \otimes^{\mathbf{L}}_R X$ is dualizing, then the previous sentence implies that $\mathbf{R}\mathrm{Hom}_R(P, R) \otimes^{\mathbf{L}}_R P \otimes^{\mathbf{L}}_R X$ is dualizing as well, because $\mathbf{R}\mathrm{Hom}_R(P, R)$ is tilting; hence, the isomorphism

$$\mathbf{R}\mathrm{Hom}_R(P, R) \otimes^{\mathbf{L}}_R P \otimes^{\mathbf{L}}_R X \simeq X$$

implies that X is dualizing.

(b) Fact 2.13 implies $P \in \mathcal{A}_C(R)$, and this explains the first isomorphism. The second isomorphism follows from Hom-tensor adjointness along with the assumption $\mathbf{R}\mathrm{Hom}_R(C, C) \simeq R$.

(c) Assume $P \simeq \mathbf{R}\mathrm{Hom}_R(B, C)$. In particular, we conclude that $\mathbf{R}\mathrm{Hom}_R(B, C)$ is semidualizing by (4.3.3), and so Theorem 1.3 implies that C is in $\mathcal{B}_B(R)$. This explains the first isomorphism in the following sequence

$$C \simeq \mathbf{R}\mathrm{Hom}_R(B, C) \otimes^{\mathbf{L}}_R B \simeq P \otimes^{\mathbf{L}}_R B$$

and the second isomorphism is by assumption. $\square$

Remark 4.9. In conjunction with the basic properties of derived tensor products, Proposition 4.8(a) yields a well-defined left $\mathrm{DPic}(R)$-action on $\overline{\mathfrak{S}}(R)$ given by $[P][C] := [P \otimes^{\mathbf{L}}_R C]$. Note that the commutativity of tensor product shows that the oppositely defined right-action is equivalent to this one.

This action yields the following relation on $\overline{\mathfrak{S}}(R)$: $[B] \approx [C]$ if $[B] = [P][C]$ for some $[P] \in \mathrm{DPic}(R)$, that is, if $[B]$ is in the orbit of $[C]$ under the $\mathrm{DPic}(R)$-action.

We conclude this section with several readily-verified properties of this relation.

Properties 4.10. Let B and C be semidualizing R-complexes.

4.10.1. For each integer n, we have $[\Sigma^n C] = [(\Sigma^n R) \otimes^{\mathbf{L}}_R C] \approx [C]$.

4.10.2. The relation $\approx$ is an equivalence relation because $\overline{\mathfrak{S}}(R)$ is partitioned into orbits by the $\mathrm{DPic}(R)$-action.

4.10.3. One has $[C] \approx [R]$ if and only if $[C] \in \mathrm{DPic}(R)$ if and only if $\mathrm{pd}_R(C) < \infty$; see Remark 4.7.

4.10.4. When R is local, we have $[B] \approx [C]$ if and only if $B \sim C$ by (4.3.4).

5. Detecting Equivalences

As we note in the introduction, the results of this section are of the following type: Assuming $\mathrm{Ext}^n_R(B,C)=0$ for all $n\gg 0$ *and a lot more*, we show $[B]\approx[C]$. The first result of this type is the following version of Property 2.11.4 which also extends Proposition 4.4 and Property 4.10.4. Note that the symmetry in conditions (i) and (ii) implies that conditions (iii) and (iv) are symmetric as well.

Proposition 5.1. *Let B and C be semidualizing R-complexes. The following conditions are equivalent:*

(i) $[B]\approx[C]$*;*
(ii) *B is C-reflexive and C is B-reflexive;*
(iii) $\mathrm{Ext}^n_R(B,C)=0$ *for all* $n\gg 0$*, and* $B_{\mathfrak{m}}\sim C_{\mathfrak{m}}$ *for each maximal ideal* $\mathfrak{m}\subset R$*;*
(iv) $\mathbf{R}\mathrm{Hom}_R(B,C)$ *is a tilting R-complex.*

When these conditions are satisfied, there are isomorphisms

$$C\simeq \mathbf{R}\mathrm{Hom}_R(B,C)\otimes^{\mathbf{L}}_R B \qquad \text{and} \qquad B\simeq \mathbf{R}\mathrm{Hom}_R(C,B)\otimes^{\mathbf{L}}_R C.$$

Proof. (i) $\Longrightarrow$ (ii). Assume $[B]\approx[C]$ and fix a tilting R-complex P such that $B\simeq P\otimes^{\mathbf{L}}_R C$. This yields the first isomorphism in each line of the following display

$$\mathbf{R}\mathrm{Hom}_R(B,C)\simeq \mathbf{R}\mathrm{Hom}_R(P\otimes^{\mathbf{L}}_R C,C)\simeq \mathbf{R}\mathrm{Hom}_R(P,R)$$
$$\mathbf{R}\mathrm{Hom}_R(C,B)\simeq \mathbf{R}\mathrm{Hom}_R(C,P\otimes^{\mathbf{L}}_R C)\simeq P$$

and the second isomorphism in each line is from Proposition 4.8(b). In particular, the complexes $\mathbf{R}\mathrm{Hom}_R(B,C)$ and $\mathbf{R}\mathrm{Hom}_R(C,B)$ are homologically bounded.

As in the proof of Proposition 4.8(a), we have $B_{\mathfrak{m}}\sim C_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}\subseteq R$. In particular, $B_{\mathfrak{m}}$ is $C_{\mathfrak{m}}$-reflexive and $C_{\mathfrak{m}}$ is $B_{\mathfrak{m}}$-reflexive for each $\mathfrak{m}$. Using the conclusion of the previous paragraph with (2.11.3) we conclude that B is C-reflexive and C is B-reflexive.

(ii) $\Longrightarrow$ (iii). Assume that B is C-reflexive and C is B-reflexive. In particular, this implies $\mathrm{Ext}^n_R(B,C)=0$ for all $n\gg 0$. It follows from (2.11.3) that $B_{\mathfrak{m}}$ is $C_{\mathfrak{m}}$-reflexive and $C_{\mathfrak{m}}$ is $B_{\mathfrak{m}}$-reflexive for each maximal ideal $\mathfrak{m}\subset R$. Hence, we have $B_{\mathfrak{m}}\sim C_{\mathfrak{m}}$ for each $\mathfrak{m}$ by (2.11.4).

(iii) $\Longrightarrow$ (iv). Assume $\mathrm{Ext}^n_R(B,C)=0$ for all $n\gg 0$, and $B_{\mathfrak{m}}\sim C_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}\subset R$. The first of these assumptions implies that $\mathbf{R}\mathrm{Hom}_R(B,C)$ is homologically finite. The second assumption yields the second isomorphism in the next sequence and the third isomorphism is from (2.9.3)

$$\mathbf{R}\mathrm{Hom}_R(B,C)_{\mathfrak{m}}\simeq \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{m}}}(B_{\mathfrak{m}},C_{\mathfrak{m}})\sim \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{m}}}(C_{\mathfrak{m}},C_{\mathfrak{m}})\simeq R_{\mathfrak{m}}.$$

Hence $\mathbf{R}\mathrm{Hom}_R(B,C)$ is tilting by Proposition 4.4.

(iv) $\Longrightarrow$ (i). This follows directly from Proposition 4.8(c).

When the conditions (i)–(iv) are satisfied, the desired isomorphisms follow from the symmetry of condition (ii) using Proposition 4.8(c). □

5.2. *Proof of Theorem 1.4.* The equivalence (i) $\Longleftrightarrow$ (ii) is in Proposition 5.1, while (ii) $\Longrightarrow$ (iii) is by definition, and (iii) $\Longrightarrow$ (iv) follows from Fact 2.16.

(iv) $\Longrightarrow$ (i). Assume that $\text{CI-dim}_R(\mathbf{R}\mathrm{Hom}_R(B,C))$ is finite. In particular, we have $\mathrm{Ext}^n_R(B,C)=0$ for all $n\gg 0$, and

$$\text{CI-dim}_{R_{\mathfrak{m}}}(\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{m}}}(B_{\mathfrak{m}},C_{\mathfrak{m}}))=\text{CI-dim}_{R_{\mathfrak{m}}}(\mathbf{R}\mathrm{Hom}_R(B,C)_{\mathfrak{m}})<\infty$$

for each maximal ideal $\mathfrak{m} \subset R$. Proposition 5.1 shows that it suffices to prove $B_\mathfrak{m} \sim C_\mathfrak{m}$ for each $\mathfrak{m}$, so we may pass to the ring $R_\mathfrak{m}$ and assume that R is local.

As $\text{CI-dim}_R(\mathbf{R}\text{Hom}_R(B,C)) < \infty$ we know from [25, (5.1.c)] that $\mathbf{R}\text{Hom}_R(B,C)$ is C-reflexive. Using Theorem 1.3, we conclude that $\mathbf{R}\text{Hom}_R(B,C)$ is semidualizing and B is C-reflexive. Proposition 4.6(a) and Property 2.9.1 imply $\mathbf{R}\text{Hom}_R(B,C) \sim R$, and this yields the second isomorphism in the next sequence

$$B \simeq \mathbf{R}\text{Hom}_R(\mathbf{R}\text{Hom}_R(B,C),C) \sim \mathbf{R}\text{Hom}_R(R,C) \simeq C.$$

The first isomorphism comes from the fact that B is C-reflexive.

(i) $\Longrightarrow$ (v). Assume $[B] \approx [C]$, and fix a tilting R-complex P and an isomorphism $\alpha\colon B \xrightarrow{\simeq} P \otimes^{\mathbf{L}}_R C$. We verify the containment $\mathcal{B}_C(R) \subseteq \mathcal{B}_B(R)$. Once we do this, the symmetry of $\approx$ implies $\mathcal{B}_B(R) \subseteq \mathcal{B}_C(R)$, thus showing $\mathcal{B}_C(R) = \mathcal{B}_B(R)$.

Let $X \in \mathcal{B}_C(R)$. The complex $\mathbf{R}\text{Hom}_R(C,X)$ is homologically bounded, and so Lemma 4.5 implies $\mathbf{R}\text{Hom}_R(C,X) \in \mathcal{B}_P(R)$. Thus, the following sequence shows that $\mathbf{R}\text{Hom}_R(B,X)$ is also homologically bounded

$$\mathbf{R}\text{Hom}_R(B,X) \simeq \mathbf{R}\text{Hom}_R(P \otimes^{\mathbf{L}}_R C, X) \simeq \mathbf{R}\text{Hom}_R(P, \mathbf{R}\text{Hom}_R(C,X)) \in \mathcal{D}_\text{b}(R).$$

To finish showing that X is in $\mathcal{B}_B(R)$, we use the next commutative diagram to conclude that ξ^B_X is an isomorphism.

$$\begin{tikzcd}
C \otimes^{\mathbf{L}}_R P \otimes^{\mathbf{L}}_R \mathbf{R}\text{Hom}_R(P, \mathbf{R}\text{Hom}_R(C,X)) \arrow[r, "C\otimes^{\mathbf{L}}_R \xi^P_{\mathbf{R}\text{Hom}_R(C,X)}", "\simeq"'] \arrow[d, "\beta\otimes^{\mathbf{L}}_R\sigma"', "\simeq"] & C \otimes^{\mathbf{L}}_R \mathbf{R}\text{Hom}_R(C,X) \arrow[d, "\simeq"', "\xi^C_X"] \\
P \otimes^{\mathbf{L}}_R C \otimes^{\mathbf{L}}_R \mathbf{R}\text{Hom}_R(P \otimes^{\mathbf{L}}_R C, X) \arrow[d, "P\otimes^{\mathbf{L}}_R C\otimes^{\mathbf{L}}_R \mathbf{R}\text{Hom}_R(\alpha,X)"', "\simeq"] & X \\
P \otimes^{\mathbf{L}}_R C \otimes^{\mathbf{L}}_R \mathbf{R}\text{Hom}_R(B,X) \arrow[r, "\alpha^{-1}\otimes^{\mathbf{L}}_R \mathbf{R}\text{Hom}_R(B,X)", "\simeq"'] & B \otimes^{\mathbf{L}}_R \mathbf{R}\text{Hom}_R(B,X) \arrow[u, "\xi^B_X"']
\end{tikzcd}$$

Here, the morphism β is tensor-commutativity, and σ is Hom-tensor adjointness. The morphisms ξ^C_X and $\xi^P_{\mathbf{R}\text{Hom}_R(C,X)}$ are isomorphisms because $X \in \mathcal{B}_C(R)$ and $\mathbf{R}\text{Hom}_R(C,X) \in \mathcal{B}_P(R)$.

(v) $\Longrightarrow$ (vi). This follows from the conditions $B \in \mathcal{B}_B(R)$ and $C \in \mathcal{B}_C(R)$.

(vi) $\Longrightarrow$ (i). Assume that $B \in \mathcal{B}_C(R)$ and $C \in \mathcal{B}_B(R)$. Theorem 1.3 implies that C is B-reflexive and B is C-reflexive, and so Proposition 5.1 implies $[B] \approx [C]$. $\square$

Our next result is a version of Theorem 1.4 for tensor products.

Proposition 5.3. *Let B and C be semidualizing R-complexes. The following conditions are equivalent:*

(i) *B and C are both tilting R-complexes, i.e, $[B] \approx [R] \approx [C]$;*
(ii) *$B \otimes^{\mathbf{L}}_R C$ is a tilting R-complex;*
(iii) $\text{pd}_R(B \otimes^{\mathbf{L}}_R C) < \infty$*;*
(iv) $\text{CI-dim}_R(B \otimes^{\mathbf{L}}_R C) < \infty$*.*

Proof. The implication (i) $\Longrightarrow$ (ii) is in Proposition 4.8(a). Also, (ii) $\Longrightarrow$ (iii) is by definition, and (iii) $\Longrightarrow$ (iv) follows from Fact 2.16.

(iv) $\Longrightarrow$ (i). Assume $\text{CI-dim}_R(B \otimes^{\mathbf{L}}_R C) < \infty$. For each maximal $\mathfrak{m} \subset R$, we have

$$\text{CI-dim}_{R_\mathfrak{m}}(B_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} C_\mathfrak{m}) = \text{CI-dim}_{R_\mathfrak{m}}((B \otimes^{\mathbf{L}}_R C)_\mathfrak{m}) < \infty.$$

We show that $B_{\mathfrak{m}} \sim R_{\mathfrak{m}} \sim C_{\mathfrak{m}}$ for each $\mathfrak{m}$, and so Proposition 4.4 implies that B and C are tilting. Thus, we may assume without loss of generality that R is local. Hence, there is a quasi-deformation $R \to R' \leftarrow Q$ such that $\mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_R (B \otimes^{\mathbf{L}}_R C))$ is finite and Q is complete; see [25, Thm. F]. Because Q is complete, Lemma A.2 implies that there are semidualizing Q-complexes M and N such that $R' \otimes^{\mathbf{L}}_Q M \simeq R' \otimes^{\mathbf{L}}_R B$ and $R' \otimes^{\mathbf{L}}_Q N \simeq R' \otimes^{\mathbf{L}}_R C$.

Each of the following isomorphisms is either standard or by assumption

$$\begin{aligned} R' \otimes^{\mathbf{L}}_R (B \otimes^{\mathbf{L}}_R C) &\simeq (R' \otimes^{\mathbf{L}}_R B) \otimes^{\mathbf{L}}_{R'} (R' \otimes^{\mathbf{L}}_R C) \\ &\simeq (R' \otimes^{\mathbf{L}}_Q M) \otimes^{\mathbf{L}}_{R'} (R' \otimes^{\mathbf{L}}_Q N) \\ &\simeq R' \otimes^{\mathbf{L}}_Q (M \otimes^{\mathbf{L}}_Q N). \end{aligned}$$

and so the finiteness of $\mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_Q (M \otimes^{\mathbf{L}}_Q N)) = \mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_R (B \otimes^{\mathbf{L}}_R C))$ implies $\mathrm{pd}_Q(M) < \infty$ and $\mathrm{pd}_Q(N) < \infty$ by Fact 2.5. Because M and N are semidualizing for Q, this means $M \sim Q \sim N$ because of (2.9.1), and the conclusion $B \sim R \sim C$ follows from Lemma A.4(a). $\square$

Comparing Theorem 1.4 and Proposition 5.3, one might feel that there is a missing condition in 5.3, namely $\mathcal{A}_B(R) = \mathcal{A}_C(R)$. The next result shows that this condition is in fact not equivalent to those in 5.3.

Proposition 5.4. *For semidualizing R-complexes B and C, one has $[B] \approx [C]$ if and only if $\mathcal{A}_B(R) = \mathcal{A}_C(R)$.*

Proof. If $[B] \approx [C]$, then argue as in the proof of the implication (i) $\implies$ (v) in Theorem 1.4 to conclude $\mathcal{A}_B(R) = \mathcal{A}_C(R)$.

For the converse, assume $\mathcal{A}_B(R) = \mathcal{A}_C(R)$, and fix a faithfully injective R-module E.[2] Observe that there is an isomorphism $\mathbf{R}\mathrm{Hom}_R(E, E) \simeq \mathrm{Hom}_R(E, E)$ because E is injective. Furthermore, the R-module $F = \mathrm{Hom}_R(E, E)$ is faithfully flat by [22, (1.5)]. In particular, an R-complex X is in $\mathcal{D}_{\mathrm{b}}(R)$ if and only if $X \otimes^{\mathbf{L}}_R F$ is in $\mathcal{D}_{\mathrm{b}}(R)$, and a morphism of R-complexes α is an isomorphism in $\mathcal{D}(R)$ if and only if $\alpha \otimes^{\mathbf{L}}_R F$ is an isomorphism in $\mathcal{D}(R)$.

We show that the containment $\mathcal{A}_B(R) \supseteq \mathcal{A}_C(R)$ implies $C \in \mathcal{B}_B(R)$. The reverse containment then yields $B \in \mathcal{B}_C(R)$, and so Theorem 1.4 implies $[B] \approx [C]$.

We know that C is in $\mathcal{B}_C(R)$, and hence [9, (2.1.f)] implies $\mathbf{R}\mathrm{Hom}_R(C, E) \in \mathcal{A}_C(R) \subseteq \mathcal{A}_B(R)$. It follows from [9, (2.1.e)] that $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, E), E)$ is in $\mathcal{B}_B(R)$. Hom-evaluation (2.6) yields the first isomorphism in the next sequence

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, E), E) \simeq C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(E, E) \simeq C \otimes^{\mathbf{L}}_R F$$

and so $C \otimes^{\mathbf{L}}_R F \in \mathcal{B}_B(R)$. From this, we know that the complex

$$\mathbf{R}\mathrm{Hom}_R(B, C \otimes^{\mathbf{L}}_R F) \simeq \mathbf{R}\mathrm{Hom}_R(B, C) \otimes^{\mathbf{L}}_R F$$

[2]Recall that E is *faithfully injective* if $\mathrm{Hom}_R(-, E)$ is faithfully exact, that is, if, for every sequence S of R-modules S is exact if and only if $\mathrm{Hom}_R(S, E)$ is exact. For example, the R-module $E = \oplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ is faithfully injective, where the sum is taken over the set of maximal ideals $\mathfrak{m} \subset R$; see [21, (3.2.2)].

is homologically bounded; the isomorphism is tensor-evaluation (2.6). As we noted above, this implies that $\mathbf{R}\mathrm{Hom}_R(B,C)$ is homologically bounded. From the following commutative diagram

$$\begin{tikzcd}
B \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B, C \otimes^{\mathbf{L}}_R F) \arrow[d, "\omega_{BCF}"', "\simeq"] \arrow[dr, "\xi^B_{C\otimes^{\mathbf{L}}_R F}", "\simeq"'] & \\
B \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B,C) \otimes^{\mathbf{L}}_R F \arrow[r, "\xi^B_C \otimes^{\mathbf{L}}_R F"'] & C \otimes^{\mathbf{L}}_R F
\end{tikzcd}$$

we conclude that $\xi^B_C \otimes^{\mathbf{L}}_R F$ is an isomorphism. As we noted above, this implies that ξ^B_C is an isomorphism, and so $C \in \mathcal{B}_B(R)$, as desired. □

The next two results are injective versions of Theorem 1.4 and Proposition 5.3. Note that we do not assume that R has a dualizing complex in either result.

Proposition 5.5. *Let B and C be semidualizing R-complexes. The following conditions are equivalent:*

(i) *B is a tilting R-complex and C is a dualizing R-complex;*
(ii) *$\mathbf{R}\mathrm{Hom}_R(B,C)$ is a dualizing R-complex;*
(iii) *$\mathrm{id}_R(\mathbf{R}\mathrm{Hom}_R(B,C)) < \infty$;*
(iv) *$\text{CI-id}_R(\mathbf{R}\mathrm{Hom}_R(B,C)) < \infty$.*

Proof. We have (ii) $\Longrightarrow$ (iii) by definition, and (iii) $\Longrightarrow$ (iv) is in Lemma A.1(a).

(i) $\Longrightarrow$ (ii). If B is tilting and C is dualizing, then $\mathrm{pd}_R(B) < \infty$ and $\mathrm{id}_R(C) < \infty$, and so [3, (4.1.I)] implies $\mathrm{id}(\mathbf{R}\mathrm{Hom}_R(B,C)) < \infty$. Also $\mathbf{R}\mathrm{Hom}_R(B,C)$ is semidualizing by (2.11.2), and hence is dualizing.

(iv) $\Longrightarrow$ (i). Assume that $\text{CI-id}_R(\mathbf{R}\mathrm{Hom}_R(B,C))$ is finite.

We first show that C is dualizing for R and $B \sim R$ when R is local. Fix a minimal generating sequence $\mathbf{x}$ for $\mathfrak{m}$, and consider the Koszul complex $K = K^R(\mathbf{x})$. As $\text{CI-id}_R(\mathbf{R}\mathrm{Hom}_R(B,C))$ and $\mathrm{pd}_R(K)$ are finite, [25, (4.4.b)] implies $\text{CI-id}_R(K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B,C)) < \infty$. The complex $K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B,C)$ has finite length homology, so [25, (3.6)] provides a quasi-deformation $R \xrightarrow{\varphi} R' \xleftarrow{\rho} Q$ such that Q is complete and $\mathrm{id}_Q(R' \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B,C)))$ is finite. By Lemmas A.2 and A.3, there are semidualizing Q-complexes M and N such that $R' \otimes^{\mathbf{L}}_Q M \simeq R' \otimes^{\mathbf{L}}_R B$ and $R' \otimes^{\mathbf{L}}_Q N \simeq R' \otimes^{\mathbf{L}}_R C$, and there is a sequence $\mathbf{y} \in Q$ such that the Koszul complex $L = K^Q(\mathbf{y})$ satisfies $R' \otimes^{\mathbf{L}}_Q L \simeq R' \otimes^{\mathbf{L}}_R K$.

In the following sequence, the first and third isomorphisms are combinations of tensor-evaluation (2.6) and Hom-tensor adjointness, and the second isomorphism is by assumption

$$\begin{aligned}
R' \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B,C) &\simeq \mathbf{R}\mathrm{Hom}_{R'}(R' \otimes^{\mathbf{L}}_R B, R' \otimes^{\mathbf{L}}_R C) \\
&\simeq \mathbf{R}\mathrm{Hom}_{R'}(R' \otimes^{\mathbf{L}}_Q M, R' \otimes^{\mathbf{L}}_Q N) \\
&\simeq R' \otimes^{\mathbf{L}}_Q \mathbf{R}\mathrm{Hom}_Q(M,N).
\end{aligned}$$

Hence, Lemma A.5 provides the following isomorphism

$$R' \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B,C)) \simeq \mathbf{R}\mathrm{Hom}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q), \mathbf{R}\mathrm{Hom}_Q(M,N)).$$

By Fact 2.5, the finiteness of

$$\begin{aligned}&\mathrm{id}_Q(R' \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(B,C)))\\ &\qquad\qquad = \mathrm{id}_Q(\mathbf{R}\mathrm{Hom}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q), \mathbf{R}\mathrm{Hom}_Q(M,N)))\end{aligned}$$

implies $\mathrm{id}_Q(\mathbf{R}\mathrm{Hom}_Q(M,N)) < \infty$ and further $\mathrm{pd}_Q(M) < \infty$ and $\mathrm{id}_Q(N) < \infty$. Because M and N are semidualizing for Q, this implies that N is dualizing for Q and $M \sim Q$. Finally, Lemma A.4 shows that C is dualizing for R and $B \sim R$. This concludes the proof when R is local.

We now show that B is a tilting R-complex and C is a dualizing R-complex in general. Our assumptions guarantee that $\mathbf{R}\mathrm{Hom}_R(B,C)$ is homologically finite, and $\text{CI-id}_{R_\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(B_\mathfrak{m}, C_\mathfrak{m})) < \infty$ for each maximal ideal $\mathfrak{m} \subset R$. The local implies $B_\mathfrak{m} \sim R_\mathfrak{m}$ and that $C_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$ for each $\mathfrak{m}$. From this, Proposition 4.4 shows that B is tilting, and so Lemma 4.5 implies $C \in \mathcal{B}_B(R)$. From Theorem 1.3 we conclude that $\mathbf{R}\mathrm{Hom}_R(B,C)$ is semidualizing and so $\mathbf{R}\mathrm{Hom}_R(B,C)$ is dualizing by Proposition 4.6(b). In the next sequence, the first isomorphism is Hom-tensor adjointness, the second isomorphism is from (4.3.1), and the third one is standard

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(B,R), \mathbf{R}\mathrm{Hom}_R(B,C)) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(B,R) \otimes^{\mathbf{L}}_R B, C)\\ &\simeq \mathbf{R}\mathrm{Hom}_R(R,C)\\ &\simeq C.\end{aligned}$$

Since $\mathrm{pd}_R(\mathbf{R}\mathrm{Hom}_R(B,R)) < \infty$ and $\mathrm{id}_R(\mathbf{R}\mathrm{Hom}_R(B,C)) < \infty$, the first complex in the above sequence has finite injective dimension by [3, (4.1.I)]. Hence $\mathrm{id}_R(C) < \infty$ and C is dualizing, as desired. Thus, we localize at $\mathfrak{m}$ in order to assume that R is local with maximal ideal $\mathfrak{m}$. □

Proposition 5.6. *Let B and C be semidualizing R-complexes. The following conditions are equivalent:*

(i) *R admits a dualizing complex D such that $B \simeq \mathbf{R}\mathrm{Hom}_R(C,D)$;*
(ii) *$B \otimes^{\mathbf{L}}_R C$ is a dualizing R-complex;*
(iii) *$\mathrm{id}_R(B \otimes^{\mathbf{L}}_R C) < \infty$;*
(iv) *$\text{CI-id}_R(B \otimes^{\mathbf{L}}_R C) < \infty$.*

Proof. We have (ii) $\Longrightarrow$ (iii) by definition, and (iii) $\Longrightarrow$ (iv) is in Lemma A.1(a). The implication (i) $\Longrightarrow$ (ii) follows from (2.11.2) using standard isomorphisms.

(iv) $\Longrightarrow$ (i). Assume that $\text{CI-id}_R(B \otimes^{\mathbf{L}}_R C)$ is finite. We show that the complex $D = B \otimes^{\mathbf{L}}_R C$ is dualizing for R; then (2.11.5) yields the desired isomorphism:

$$\mathbf{R}\mathrm{Hom}_R(C,D) = \mathbf{R}\mathrm{Hom}_R(C, B \otimes^{\mathbf{L}}_R C) \simeq B.$$

The finiteness of $\text{CI-id}_R(B \otimes^{\mathbf{L}}_R C)$ implies that $B \otimes^{\mathbf{L}}_R C$ is homologically finite, and $\text{CI-dim}_{R_\mathfrak{m}}(B_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} C_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m} \subset R$. We show how this implies that the complex $(B \otimes^{\mathbf{L}}_R C)_\mathfrak{m} \simeq B_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} C_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$ for each $\mathfrak{m}$. From this, Lemma A.1(b) implies $\mathrm{id}_R(B \otimes^{\mathbf{L}}_R C) < \infty$, and (2.9.3) shows that $B \otimes^{\mathbf{L}}_R C$ is semidualizing for R; hence $B \otimes^{\mathbf{L}}_R C$ is dualizing. Thus, we localize at $\mathfrak{m}$ in order to assume that R is local with maximal ideal $\mathfrak{m}$.

The assumption $\text{CI-id}_R(B \otimes^{\mathbf{L}}_R C) < \infty$ implies that $B \otimes^{\mathbf{L}}_R C$ is homologically finite, and so $B \otimes^{\mathbf{L}}_R C \in \mathcal{B}_C(R)$ by [25, (5.2.b)]. Fact 2.13 then implies $B \in \mathcal{A}_C(R)$, and so $B \otimes^{\mathbf{L}}_R C$ is semidualizing by Corollary 3.8. It follows from Proposition 4.6(b) that $B \otimes^{\mathbf{L}}_R C$ is dualizing as desired. □

6. Ext-vanishing, Tor-vanishing and Cohen-Macaulayness

The following is proved in [2, (1.3)] and serves as our motivation for this section.

Fact 6.1. Let R be a local ring admitting a dualizing complex D and assume $\inf(D) = 0$. If $\mathrm{Ext}^n_R(D, R) = 0$ for $i = 1, \ldots, \dim(R)$, then R is Cohen-Macaulay.

We generalize this fact to the realm of semidualizing complexes after the following lemma which compliments (2.9.7).

Lemma 6.2. *Let R be a local ring and C a semidualizing R-complex. If* $\mathrm{amp}(C) = \mathrm{cmd}(R)$ *and* $\mathrm{Supp}_R(\mathrm{H}_{\sup(C)}(C)) = \mathrm{Spec}(R)$*, then C is Cohen-Macaulay.*

Proof. After replacing C with $\mathsf{\Sigma}^{-\inf(C)}C$, we assume without loss of generality that $\inf(C) = 0$, which implies $\sup(C) = \mathrm{amp}(C) = \mathrm{cmd}(R)$. We show that, for each $\mathfrak{p} \in \mathrm{Spec}(R)$, we have $\dim(R/\mathfrak{p}) - \inf(C_\mathfrak{p}) \leqslant \mathrm{depth}_R(C)$. From this it follows that

$$\mathrm{depth}_R(C) \leqslant \dim_R(C) = \sup\{\dim(R/\mathfrak{p}) - \inf(C_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R)\} \leqslant \mathrm{depth}_R(C)$$

and so C is Cohen-Macaulay.

Fix a prime $\mathfrak{p} \in \mathrm{Spec}(R)$. Our assumptions yield the following sequence

$$0 = \inf(C) \leqslant \inf(C_\mathfrak{p}) \leqslant \sup(C_\mathfrak{p}) = \sup(C) = \mathrm{cmd}(R).$$

The complex $C_\mathfrak{p}$ is semidualizing for $R_\mathfrak{p}$ and so (2.9.7) implies

$$\mathrm{cmd}(R) - \inf(C_\mathfrak{p}) = \mathrm{amp}(C_\mathfrak{p}) \leqslant \mathrm{cmd}(R_\mathfrak{p}).$$

This explains the first inequality in the following sequence

$$\begin{aligned}
\dim(R/\mathfrak{p}) - \inf(C_\mathfrak{p}) &\leqslant \dim(R/\mathfrak{p}) - \mathrm{cmd}(R) + \mathrm{cmd}(R_\mathfrak{p}) \\
&= [\dim(R/\mathfrak{p}) + \dim(R_\mathfrak{p}) - \dim(R)] + \mathrm{depth}(R) - \mathrm{depth}(R_\mathfrak{p}) \\
&\leqslant \mathrm{depth}(R) - \mathrm{depth}(R_\mathfrak{p}) \\
&\leqslant \mathrm{depth}(R) \\
&= \mathrm{depth}_R(C).
\end{aligned}$$

The first equality is by definition; the second and third inequalities are standard; and the final equality is from [8, (3.2.a)]. $\square$

See Remark 6.5 for an explicit discussion of the connection between Fact 6.1 and the next result.

Proposition 6.3. *Let R be a local ring and fix semidualizing R-complexes B and C such that* $\inf(B) = 0 = \inf(C)$*.*

(a) *Fix a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R_\mathfrak{p}$ is Cohen-Macaulay, e.g., $\mathfrak{p} \in \mathrm{Min}(R)$. If $i = \sup(B_\mathfrak{p})$ and $j = \sup(C_\mathfrak{p})$, then $\mathrm{Ext}^{i-j}_R(B, C) \neq 0$.*
(b) *Fix an integer $s \geqslant \sup(C)$. If $\mathrm{Ext}^n_R(B, C) = 0$ for $n = -s+1, \ldots, \dim(R)$, then B is isomorphic to a module in $\mathcal{D}(R)$, $s = \sup(C)$ and* $\mathrm{Supp}_R(\mathrm{H}_s(C)) = \mathrm{Spec}(R)$*.*
(c) *If $\mathrm{Ext}^n_R(B, C) = 0$ for $n = -\mathrm{cmd}(R) + 1, \ldots, \dim(R)$, then C is Cohen-Macaulay.*
(d) *Assume that R admits a dualizing complex D such that* $\inf(D) = 0$ *and that* $\sup(C) = 0$*. If $\mathrm{Ext}^n_R(\mathbf{R}\mathrm{Hom}_R(C, D), C) = 0$ for $n = 1, \ldots, \dim(R)$, then R is Cohen-Macaulay.*

(e) *Assume that R admits a dualizing complex D such that $\inf(D) = 0$ and that B is Cohen-Macaulay. If $\operatorname{Ext}^n_R(B, \mathbf{R}\operatorname{Hom}_R(B, D)) = 0$ for $n = 1, \dots, \dim(R)$, then R is Cohen-Macaulay.*

Proof. (a) The equality $\operatorname{Supp}_R(B) = \operatorname{Spec}(R)$ from (2.9.5) implies $0 \leqslant i \leqslant \sup(B)$, and similarly $0 \leqslant j \leqslant \sup(C)$. Furthermore, since $R_\mathfrak{p}$ is Cohen-Macaulay, we have $\operatorname{amp}(B_\mathfrak{p}) = 0 = \operatorname{amp}(C_\mathfrak{p})$ by (2.9.7). In particular, there are isomorphisms

$$B_\mathfrak{p} \simeq \Sigma^i \operatorname{H}_i(B)_\mathfrak{p} \qquad\qquad C_\mathfrak{p} \simeq \Sigma^j \operatorname{H}_j(C)_\mathfrak{p}$$

and these provide the second isomorphism in the next sequence

$$\begin{aligned} \operatorname{Ext}^{i-j}_R(B, C)_\mathfrak{p} &\cong \operatorname{Ext}^{i-j}_{R_\mathfrak{p}}(B_\mathfrak{p}, C_\mathfrak{p}) \cong \operatorname{Ext}^{i-j}_{R_\mathfrak{p}}(\Sigma^i \operatorname{H}_i(B)_\mathfrak{p}, \Sigma^j \operatorname{H}_j(C)_\mathfrak{p}) \\ &\cong \operatorname{Ext}^0_{R_\mathfrak{p}}(\operatorname{H}_i(B)_\mathfrak{p}, \operatorname{H}_j(C)_\mathfrak{p}) \cong \operatorname{Hom}_{R_\mathfrak{p}}(\operatorname{H}_i(B)_\mathfrak{p}, \operatorname{H}_j(C)_\mathfrak{p}) \neq 0. \end{aligned}$$

The first, third and fourth isomorphisms are standard; and the nonvanishing holds by [20, (3.6)] because $\operatorname{H}_i(B)_\mathfrak{p}$ is a semidualizing $R_\mathfrak{p}$-module. We conclude that $\operatorname{Ext}^{i-j}_R(B, C) \neq 0$, as desired.

(b) Fix a prime ideal $\mathfrak{p} \in \operatorname{Min}(R)$, and set $i = \sup(B_\mathfrak{p})$ and $j = \sup(C_\mathfrak{p})$. The assumption $\inf(B) = 0 = \inf(C)$ implies $0 \leqslant i \leqslant \sup(B)$ and $0 \leqslant j \leqslant \sup(C) \leqslant s$. This justifies the first five inequalities in the next sequence

$$-s \leqslant -\sup(C) \leqslant -j \leqslant i - j \leqslant i \leqslant \sup(B) = \operatorname{amp}(B) \leqslant \operatorname{cmd}(R) \leqslant \dim(R).$$

The sixth inequality is in (2.9.7), and the equality is by assumption.

From part (a) we know $\operatorname{Ext}^{i-j}_R(B, C) \neq 0$. Hence, the previously displayed inequalities along with our vanishing hypothesis imply

$$-s = -\sup(C) = -j = i - j$$

Hence, we have $j = \sup(C) = s$ and $i = 0$. The first equality with the definition $j = \sup(C_\mathfrak{p})$ implies $\mathfrak{p} \in \operatorname{Supp}_R(\operatorname{H}_s(C))$. Since $\mathfrak{p}$ is an arbitrary prime in $\operatorname{Min}(R)$, it follows that $\operatorname{Min}(R) \subseteq \operatorname{Supp}_R(\operatorname{H}_s(C))$. Because $\operatorname{Supp}_R(\operatorname{H}_s(C))$ is Zariski closed in $\operatorname{Spec}(R)$, the equality $\operatorname{Supp}_R(\operatorname{H}_s(C)) = \operatorname{Spec}(R)$ follows.

As in the previous paragraph, the equality $0 = i = \sup(B_\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Min}(R)$ implies $\operatorname{Supp}_R(\operatorname{H}_0(B)) = \operatorname{Spec}(R)$. In particular, for each $\mathfrak{q} \in \operatorname{Spec}(R)$, we have $\inf(B_\mathfrak{q}) = 0$. For each $\mathfrak{q} \in \operatorname{Ass}_R(\operatorname{H}_{\sup(B)}(B))$ this yields the second equality in the next sequence

$$\sup(B) = \inf(B_\mathfrak{q}) = 0.$$

The first equality is from (2.9.3). Thus B is isomorphic to a module in $\mathcal{D}(R)$.

(c) Since $\operatorname{cmd}(R) \geqslant \operatorname{amp}(C) = \sup(C)$ by (2.9.7), part (b) implies $\sup(C) = \operatorname{cmd}(R)$ and $\operatorname{Supp}_R(\operatorname{H}_{\sup(C)}(C)) = \operatorname{Spec}(R)$. Now apply Lemma 6.2 to conclude that C is Cohen-Macaulay.

(d) From (2.11.2) we know $\inf(\mathbf{R}\operatorname{Hom}_R(C, D)) = 0$, and so part (b) provides the first equality in the next sequence

$$0 = \operatorname{amp}((\mathbf{R}\operatorname{Hom}_R(C, D)) = \operatorname{cmd}(C) = \operatorname{cmd}(R).$$

The remaining equalities are from (2.11.2) and (2.9.7), respectively, using the assumption $\operatorname{amp}(C) = 0$. Hence, R is Cohen-Macaulay, as desired.

(e) If B is Cohen-Macaulay, then $\operatorname{amp}(\mathbf{R}\operatorname{Hom}_R(B, D)) = 0$ by (2.11.2). The isomorphism $\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(B, D), D) \simeq B$ shows that we may apply part (d) using the complex $C = \mathbf{R}\operatorname{Hom}_R(B, D)$ to conclude that R is Cohen-Macaulay. $\square$

The next remark shows the need for the hypotheses in Proposition 6.3(d) and (e).

Remark 6.4. Let R be a local ring and fix semidualizing R-complexes B and C such that $\inf(B) = 0 = \inf(C)$ and $\operatorname{Ext}^n_R(B, C) = 0$ for $n = 1, \ldots, \dim(R)$. In view of Fact 6.1 and Proposition 6.3, one might be tempted to conclude that R is Cohen-Macaulay. However, the example $B = R = C$ shows that this conclusion need not follow. Using the same B and C with Proposition 6.6 in mind, we see that the assumption $\operatorname{Tor}^R_n(B, C) = 0$ for $n = 1, \ldots, 2\dim(R)$ also does not necessarily imply that R is Cohen-Macaulay.

We next explain why Fact 6.1 is a special case of Proposition 6.3.

Remark 6.5. Let R be a local ring, and let D be a dualizing complex for R such that $\inf(D) = 0$ and $\operatorname{Ext}^n_R(D, R) = 0$ for $i = 1, \ldots, \dim(R)$. Notice that D is Cohen-Macaulay. Apply Proposition 6.3(b) with $B = D$ and $C = R$ to conclude $\sup(D) = 0$. Hence, the equality $\sup(D) = \operatorname{cmd}(R)$ implies that R is Cohen-Macaulay. One can also apply Proposition 6.3(d) with $C = R$ or Proposition 6.3(e) with $B = D$ to draw the same conclusion.

The proof of the following is very similar to that of Proposition 6.3.

Proposition 6.6. *Let R be a local ring and fix semidualizing R-complexes B and C such that $\inf(B) = 0 = \inf(C)$. Set $r = \sup(B)$ and $s = \sup(C)$.*

(a) *If $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{H}_r(B))$ and $j = \inf(C_{\mathfrak{p}})$ then $\operatorname{Tor}^R_{r+j}(B, C) \neq 0$.*
(b) *If $\operatorname{Tor}^R_n(B, C) = 0$ for $n = 1, \ldots, 2\dim(R)$, then B and C are isomorphic to modules in $\mathcal{D}(R)$.*
(c) *Assume that either B or C is Cohen-Macaulay. If $\operatorname{Tor}^R_n(B, C) = 0$ for $n = 1, \ldots, 2\dim(R)$, then R is Cohen-Macaulay.* $\square$

7. Special Cases of Question 1.2(a)

We open this section with an example showing why we need R to be local in Questions 1.2(a) and 3.9(a).

Example 7.1. Let $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ be commutative noetherian local rings, and set $R = R_1 \times R_2$. Assume that R_2 is Cohen-Macaulay and non-Gorenstein and admits a dualizing module D_2. Fix an integer $m > \dim(R) + 1$, and consider the R-complex $B = R_1 \oplus \Sigma^m D_2$.

Set $\mathfrak{m}' = \mathfrak{m}_1 \times R_2$ and $\mathfrak{m}'' = R_1 \times \mathfrak{m}_2$. Recall that $\{\mathfrak{m}', \mathfrak{m}''\}$ is precisely the set of maximal ideals of R. Also, there are isomorphisms

$$\begin{aligned} R_{\mathfrak{m}'} &\cong (R_1)_{\mathfrak{m}_1} \cong R_1 & R_{\mathfrak{m}''} &\cong (R_2)_{\mathfrak{m}_2} \cong R_2 \\ B_{\mathfrak{m}'} &\simeq (R_1)_{\mathfrak{m}_1} \simeq R_1 & B_{\mathfrak{m}''} &\simeq \Sigma^m (D_2)_{\mathfrak{m}_2} \simeq \Sigma^m D_2. \end{aligned}$$

These isomorphisms with (2.9.3) imply that B is semidualizing for R.

We claim that $\operatorname{Ext}^n_R(B, R) = 0$ for $n = 1, \ldots, m-1$. Indeed, the above isomorphisms imply

$$\begin{aligned} \operatorname{Ext}^n_R(B, R)_{\mathfrak{m}'} &\cong \operatorname{Ext}^n_{R_{\mathfrak{m}'}}(B_{\mathfrak{m}'}, R_{\mathfrak{m}'}) \cong \operatorname{Ext}^n_{R_1}(R_1, R_1) = 0 && \text{for } n \geqslant 1 \\ \operatorname{Ext}^n_R(B, R)_{\mathfrak{m}''} &\cong \operatorname{Ext}^n_{R_{\mathfrak{m}''}}(B_{\mathfrak{m}''}, R_{\mathfrak{m}''}) \\ &\cong \operatorname{Ext}^n_{R_2}(\Sigma^m D_2, R_2) \cong \operatorname{Ext}^{n-m}_{R_2}(D_2, R_2) = 0 && \text{for } n < m. \end{aligned}$$

This establishes the claim since it can be verified locally.

Suppose that B is R-reflexive. We conclude from (2.11.3) that $B_{\mathfrak{m}''}$ is $R_{\mathfrak{m}''}$-reflexive, that is D_2 is R_2-reflexive. From (2.11.2) we know that R_2 is D_2-reflexive,

so (2.11.4) implies $D_2 \sim R_2$, contradicting the assumption that R_2 is non-Gorenstein; see (2.9.2). This shows why we must assume that R is local in Question 1.2(a).

For Question 3.9(a) argue as above to conclude that $\operatorname{Tor}^n_R(B,B) = 0$ for $n = 1,\ldots,2m-1$. Suppose that $B \otimes^{\mathbf{L}}_R B$ is semidualizing for R. From (2.9.3) we conclude that that $(B \otimes^{\mathbf{L}}_R B)_{\mathfrak{m}''}$ is semidualizing for $R_{\mathfrak{m}''}$, that is $D_2 \otimes^{\mathbf{L}}_{R_2} D_2$ is semidualizing for R_2. Using [15, (3.2)] we conclude $D_2 \sim R_2$, again contradicting the assumption that R_2 is non-Gorenstein.

7.2. *Proof of Theorem 1.5.* In each case, Proposition 6.3 implies that R is Cohen-Macaulay. In particular (2.9.7) implies

$$\sup(B) = \sup(C) = \sup(D) = 0 = \operatorname{amp}(\mathbf{R}\operatorname{Hom}_R(B,D)) = \operatorname{amp}(\mathbf{R}\operatorname{Hom}_R(C,D)).$$

Hence, we assume that B, C and D are modules. From (2.11.2) there are equalities $\inf(\mathbf{R}\operatorname{Hom}_R(B,D)) = \inf(\mathbf{R}\operatorname{Hom}_R(C,D)) = 0$ and this yields isomorphisms

$$\mathbf{R}\operatorname{Hom}_R(B,D) \simeq \operatorname{Hom}_R(B,D) \qquad \text{and} \qquad \mathbf{R}\operatorname{Hom}_R(C,D) \simeq \operatorname{Hom}_R(C,D).$$

(a) We claim that there is an isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}_R(B,\operatorname{Hom}_R(B,D)),D) \cong B \otimes_R B. \tag{7.2.1}$$

The argument is akin to that of [2, (B.3)]. Let $D \xrightarrow{\simeq} J$ be a minimal injective resolution, and set $G = \mathbf{R}\operatorname{Hom}_R(B,\operatorname{Hom}_R(B,D))$. In particular, we have $J_p = 0$ when $p \geqslant 1$ and when $p < -\dim(R)$. From [2, (B.1)], there is a strongly convergent spectral sequence

$$E^2_{p,q} = \mathrm{H}_p(\operatorname{Hom}_R(\mathrm{H}_{-q}(G),J)) \implies \mathrm{H}_{p+q}(\operatorname{Hom}_R(G,J)).$$

The differentials act in the pattern $d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$.

Our vanishing assumptions for $\operatorname{Ext}^q_R(B,\operatorname{Hom}_R(B,D)) \cong \mathrm{H}_{-q}(G)$ and J_p imply $E^2_{p,q} = 0$ when $1 \leqslant q \leqslant \dim(R)$, when $p \geqslant 1$ and when $p < -\dim(R)$. In particular, we have $E^\infty_{-q,q} = 0$ when $q \neq 0$ and

$$E^\infty_{0,0} = E^2_{0,0} = \mathrm{H}_0(\operatorname{Hom}_R(\mathrm{H}_0(G),J)) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(B,\operatorname{Hom}_R(B,D)),D).$$

The isomorphism is by the left-exactness of $\operatorname{Hom}_R(\operatorname{Hom}_R(B,\operatorname{Hom}_R(B,D)),-)$. The vanishing $E^\infty_{-q,q} = 0$ when $q \neq 0$ yields the first isomorphism in the following

$$\mathrm{H}_0(\operatorname{Hom}_R(G,J)) \cong E^\infty_{0,0} \cong \operatorname{Hom}_R(\operatorname{Hom}_R(B,\operatorname{Hom}_R(B,D)),D). \tag{7.2.2}$$

On the other hand, the first isomorphism in the next sequence is by construction

$$\begin{aligned} \operatorname{Hom}_R(G,J) &\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(B,\operatorname{Hom}_R(B,D)),D) \\ &\simeq B \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(B,D),D) \\ &\simeq B \otimes^{\mathbf{L}}_R B. \end{aligned}$$

The second isomorphism is by Hom-evaluation (2.6), and the third is from (2.11.2). Taking degree-0 homology in this sequence and using Fact 2.4, we have

$$\mathrm{H}_0(\operatorname{Hom}_R(G,J)) \cong \mathrm{H}_0(B \otimes^{\mathbf{L}}_R B) \cong B \otimes_R B.$$

With (7.2.2), this provides the isomorphism (7.2.1).

The isomorphism (7.2.1) yields the first equality in the following sequence

$$\begin{aligned} \operatorname{Ass}_R(B \otimes_R B) &= \operatorname{Ass}_R(\operatorname{Hom}_R(\operatorname{Hom}_R(B,\operatorname{Hom}_R(B,D)),D)) \\ &= \operatorname{Supp}_R(\operatorname{Hom}_R(B,\operatorname{Hom}_R(B,D))) \cap \operatorname{Ass}_R(D) \\ &\subseteq \operatorname{Ass}_R(R). \end{aligned}$$

The second equality is standard, and the containment is by (2.9.6). Now use [2, (2.2)] to conclude $B \cong R$.

(b) Because $\operatorname{amp}(C) = 0$, we know that $\mathbf{R}\mathrm{Hom}_R(C, D)$ is Cohen-Macaulay by (2.11.2). Since $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, D), D) \simeq C$ we may apply part (a) with $B = \mathbf{R}\mathrm{Hom}_R(C, D)$ to conclude $\mathbf{R}\mathrm{Hom}_R(C, D) \simeq R$ and hence

$$C \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C, D), D) \simeq \mathbf{R}\mathrm{Hom}_R(R, D) \simeq D. \qquad \square$$

Remark 7.3. It is natural to try to use Theorem 1.5 to answer Question 3.9 when R is generically Gorenstein. To see what goes wrong, let R, B, C, and D be as in Theorem 1.5, and assume that $\mathrm{Tor}^R_n(B, C) = 0$ for $n = 1, \ldots, d$ for some integer d. Our Tor-vanishing hypothesis combines with the isomorphism

$$\mathbf{R}\mathrm{Hom}_R(B \otimes^{\mathbf{L}}_R C, D) \simeq \mathbf{R}\mathrm{Hom}_R(B, \mathbf{R}\mathrm{Hom}_R(C, D))$$

to imply $\mathrm{Ext}^n_R(B, \mathbf{R}\mathrm{Hom}_R(C, D)) = 0$ for $d - \dim(R)$ consecutive values of $n \geqslant 1$. However, we cannot apply Theorem 1.5 to B and $\mathbf{R}\mathrm{Hom}_R(C, D)$ even when d is large, because the Ext-vanishing begins at $n = \dim(R) + 1$, not at $n = 1$.

To highlight the difficulties outlined in the previous remark, we pose the following question which is a special case of Question 3.9(b).

Question 7.4. Assume that R is a complete Cohen-Macaulay local ring of dimension $d \geqslant 1$ that is Gorenstein on the punctured spectrum. Let B and C be semidualzing R-modules such that $\mathrm{Tor}^R_n(B, C) = 0$ for all $n \geqslant 1$. Must $B \otimes^{\mathbf{L}}_R C$ be semidualizing for R?

In the next result, the special case $B = D$ in part (a) (or $C = R$ in part (b)) is exactly [2, (3.1)]. Note that, if Q admits a dualizing complex, that is, if Q is a homomorphic image of a Gorenstein ring, or if Q is excellent, then Q automatically has Gorenstein formal fibres.

Proposition 7.5. *Let Q be a generically Gorenstein, local ring with Gorenstein formal fibres. Let $\mathbf{x} = x_1, \ldots, x_c \in Q$ be a Q-regular sequence, and set $R = Q/(\mathbf{x})$. Assume that R admits a dualizing complex D such that* $\inf(D) = 0$*. Let B and C be semidualizing R-complexes such that* $\inf(B) = 0 = \inf(C)$*. Assume that B is Cohen-Macaulay and* $\sup(C) = 0$*.*

(a) *If $\mathrm{Ext}^n_R(B, \mathbf{R}\mathrm{Hom}_R(B, D)) = 0$ for $n = 1, \ldots, \dim(R) + 1$, then $B \simeq R$.*
(b) *If $\mathrm{Ext}^n_R(\mathbf{R}\mathrm{Hom}_R(C, D), C) = 0$ for $n = 1, \ldots, \dim(R) + 1$, then $C \simeq D$.*

Proof. In each case, Proposition 6.3 implies that R is Cohen-Macaulay and hence, so is Q. As in the proof of Theorem 1.5 we assume that B, C and D are modules and we have $\mathbf{R}\mathrm{Hom}_R(B, D) \simeq \mathrm{Hom}_R(B, D)$ and $\mathbf{R}\mathrm{Hom}_R(C, D) \simeq \mathrm{Hom}_R(C, D)$.

(a) We first reduce to the case where $c = 1$, as in the proof of [2, (3.1)]. From [2, (3.2)] there is a Q-regular sequence $\mathbf{y} \in Q$ such that $(\mathbf{y})Q = (\mathbf{x})Q$ and $P = Q/(y_1, \ldots, y_{c-1})$ is generically Gorenstein. Note that P has Gorenstein formal fibres because Q does, and so we may replace Q with P in order to assume $c = 1$.

Next, we reduce to the case where Q (and hence R) is complete. The sequence $\mathbf{x} = x_1$ is $\widehat{Q}$-regular because $\widehat{Q}$ is Q-flat, and there is an isomorphism $\widehat{R} \cong \widehat{Q}/(x_1)$. The ring $\widehat{R}$ is Cohen-Macaulay with dualizing module $\widehat{R} \otimes_R D$ by (2.9.4). Furthermore, the module $\widehat{R} \otimes_R B \simeq \widehat{R} \otimes^{\mathbf{L}}_R B$ is $\widehat{Q}$-semidualizing by (2.9.4). The standard

base-change result for flat extensions yields the following isomorphisms

$$\begin{aligned}\operatorname{Ext}^n_{\widehat{R}}(\widehat{R}\otimes_R B, \operatorname{Hom}_{\widehat{R}}(\widehat{R}\otimes_R B, \widehat{R}\otimes_R D)) &\cong \operatorname{Ext}^n_{\widehat{R}}(\widehat{R}\otimes_R B, \widehat{R}\otimes_R \operatorname{Hom}_R(B,D)) \\ &\cong \widehat{R}\otimes_R \operatorname{Ext}^n_R(B, \operatorname{Hom}_R(B,D))\end{aligned}$$

for each integer n. In particular, we have $\operatorname{Ext}^n_{\widehat{R}}(\widehat{R}\otimes_R B, \operatorname{Hom}_{\widehat{R}}(\widehat{R}\otimes_R B, \widehat{R}\otimes_R D)) = 0$ for $i = 1,\dots,\dim(R)+1$, that is, $i = 1,\dots,\dim(\widehat{R})+1$. If it follows that $\widehat{R}\otimes_R B \cong \widehat{R}$, then (2.9.4) implies $B \cong R$, as desired.

To finish the reduction, we need to explain why $\widehat{Q}$ is generically Gorenstein. Because $\widehat{Q}$ is Cohen-Macaulay, there is an equality $\operatorname{Ass}(\widehat{Q}) = \operatorname{Min}(\widehat{Q})$. Fix a prime ideal $\mathfrak{q} \in \operatorname{Min}(\widehat{Q})$ and set $\mathfrak{p} = \mathfrak{q}\cap Q$. The going-down-property implies $\mathfrak{p} \in \operatorname{Min}(Q)$, and so $Q_{\mathfrak{p}}$ is Gorenstein by assumption. The induced morphism $Q_{\mathfrak{p}} \to \widehat{Q}_{\mathfrak{q}}$ is flat and local, and the fact that Q has Gorenstein formal fibres implies that this map has Gorenstein closed fibre. It follows that $\widehat{Q}_{\mathfrak{q}}$ is Gorenstein, as desired.

Now, we assume that Q and R are complete, so the ring Q admits a dualizing module D^Q. Lemma A.2 implies that Q admits a semidualizing complex A such that $B \simeq R\otimes^{\mathbf{L}}_Q A$, and (2.9.4) implies $\operatorname{amp}(A) = \operatorname{amp}(B) = 0$. That is, A is a semidualizing Q-module such that $B \cong R\otimes_Q A$. (This is also proved by Gerko [17, (3)].) Note that x_1 is A-regular and D^Q-regular by (2.9.6), and also $D \cong R\otimes_Q D^Q$. By [2, (3.3.1)] the hypothesis $\operatorname{Ext}^n_R(B, \operatorname{Hom}_R(B,D)) = 0$ for $i = 1,\dots,\dim(R)+1$ implies $\operatorname{Ext}^n_P(A, \operatorname{Hom}_R(A, D^Q)) = 0$ for $i = 1,\dots,\dim(Q)$, and so Theorem 1.5 implies $A \cong Q$. It follows that $B \cong R\otimes_Q A \cong R\otimes_Q Q \cong R$, as desired.

(b) This follows from part (a) as in the proof of Theorem 1.5(b). □

Our final result follows from Proposition 7.5 using the argument of [2, (4.1), (4.2)] which is, in turn, the special case $B = D$ in part (a) (or $C = R$ in part (b)).

Proposition 7.6. *Let R be a local ring of the form $Q/\mathfrak{a}$, where Q is a Gorenstein local ring and $\mathfrak{a}$ is an ideal for which there is a sequence of links $\mathfrak{a} \sim \mathfrak{b}_1 \sim \cdots \sim \mathfrak{b}_s \sim \mathfrak{b}$ with $\mathfrak{b}$ a generically complete intersection ideal. (This is the case, e.g., if R is a homomorphic image of a Gorenstein ring and* $\operatorname{codepth}(R) \leqslant 2$.*) Let D be a dualizing R-complex such that* $\inf(D) = 0$, *and let B and C be semidualizing R-complexes such that* $\inf(B) = 0 = \inf(C)$. *Assume that B is Cohen-Macaulay and* $\sup(C) = 0$.

(a) *If* $\operatorname{Ext}^n_R(B, \mathbf{R}\operatorname{Hom}_R(B,D)) = 0$ *for $n = 1,\dots,\dim(R)+1$, then $B \simeq R$.*
(b) *If* $\operatorname{Ext}^n_R(\mathbf{R}\operatorname{Hom}_R(C,D), C) = 0$ *for $n = 1,\dots,\dim(R)+1$, then $C \simeq D$.* □

Appendix A. Complete Intersection Dimensions and Quasi-deformations

This appendix contains several technical lemmas for use in Sections 4 and 5.

Lemma A.1. *Let X be a homologically finite R-complex.*

(a) *There is an inequality* $\operatorname{CI-id}_R(X) \leqslant \operatorname{id}_R(X)$ *with equality when* $\operatorname{id}_R(X) < \infty$.
(b) *If* $\operatorname{CI-id}_R(X) < \infty$ *and* $\operatorname{id}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) < \infty$ *for each maximal ideal $\mathfrak{m} \subset R$, then* $\operatorname{id}_R(X) = \operatorname{CI-id}_R(X) < \infty$.

Proof. Assume that $\operatorname{id}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) < \infty$ for every maximal ideal $\mathfrak{m} \subset R$. Fact 2.16 shows $\operatorname{CI-id}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) < \infty$ for each $\mathfrak{m}$, and so the following Bass formulas come from [25, (2.11)] and [12, (2.7.b)]

$$\operatorname{CI-id}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) = \operatorname{depth}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}) - \inf(X_{\mathfrak{m}}) = \operatorname{id}_{R_{\mathfrak{m}}}(X_{\mathfrak{m}}).$$

This yields the second equality in the next sequence

$$\begin{aligned}\mathrm{id}_R(X) &= \sup\{\mathrm{id}_{R_\mathfrak{m}}(X_\mathfrak{m}) \mid \mathfrak{m} \subset R \text{ is a maximal ideal}\}\\ &= \sup\{\text{CI-id}_{R_\mathfrak{m}}(X_\mathfrak{m}) \mid \mathfrak{m} \subset R \text{ is a maximal ideal}\}\\ &= \text{CI-id}_R(X).\end{aligned}$$

The first equality is from [3, (5.3.I)], and the third equality is in Definition 2.15. Part (b) follows immediately.

(a) Assume that $\mathrm{id}_R(X) < \infty$. This implies $\mathrm{id}_{R_\mathfrak{m}}(X_\mathfrak{m}) < \infty$ for every maximal $\mathfrak{m} \subset R$ by [3, (5.3.I)], so the desired conclusion follows from the previous display. $\square$

Lemma A.2. *Assume that R is local and let $R \to R' \leftarrow Q$ be a quasi-deformation such that Q is complete. For each semidualizing R-complex C there exists a semidualizing Q-complex N such that $R' \otimes^{\mathbf{L}}_Q N \simeq R' \otimes^{\mathbf{L}}_R C$.*

Proof. The complex $R' \otimes^{\mathbf{L}}_R C$ is semidualizing for R' by (2.9.4). In particular, this implies $\mathrm{Ext}^2_{R'}(R' \otimes^{\mathbf{L}}_R C, R' \otimes^{\mathbf{L}}_R C) = 0$. Thus, because Q is complete and the map $Q \to R'$ is surjective with kernel generated by a Q-sequence, a lifting result of Yoshino [30, (3.2)] provides a homologically finite Q-complex N such that $R' \otimes^{\mathbf{L}}_Q N \simeq R' \otimes^{\mathbf{L}}_R C$. Again using the fact that $R' \otimes^{\mathbf{L}}_R C$ is semidualizing for R', this isomorphism implies that N is semidualizing for Q by [14, (4.5)]. $\square$

In the next result, the Koszul complex over R on a sequence $\mathbf{x}$ is denoted $K^R(\mathbf{x})$.

Lemma A.3. *Assume that R is local and let $R \xrightarrow{\varphi} R' \xleftarrow{\rho} Q$ be a quasi-deformation. For each sequence $\mathbf{x} = x_1, \ldots, x_n \in R$ there exists a sequence $\mathbf{y} = y_1, \ldots, y_n \in Q$ such that $R' \otimes^{\mathbf{L}}_R K^R(\mathbf{x}) \simeq R' \otimes^{\mathbf{L}}_Q K^Q(\mathbf{y})$.*

Proof. Because ρ is surjective, there exist elements $y_i \in Q$ such that $\rho(y_i) = \varphi(x_i)$ for each i. The desired isomorphism now follows from the next sequence

$$R' \otimes^{\mathbf{L}}_R K^R(\mathbf{x}) \simeq K^{R'}(\phi(\mathbf{x})) \simeq K^{R'}(\rho(\mathbf{y})) \simeq R' \otimes^{\mathbf{L}}_Q K^Q(\mathbf{y}). \qquad \square$$

Lemma A.4. *Assume that R is local and let $R \to R' \leftarrow Q$ be a quasi-deformation. Let C be a semidualzing R-complex and assume that N is a semidualzing Q-complex such that $R' \otimes^{\mathbf{L}}_Q N \simeq R' \otimes^{\mathbf{L}}_R C$.*

(a) *If $N \sim Q$, then $C \sim R$.*
(b) *If N is dualizing for Q, then C is dualizing for R.*

Proof. (a) Assuming $N \sim Q$, the next isomorphisms follow from our hypotheses

$$R' \otimes^{\mathbf{L}}_R C \simeq R' \otimes^{\mathbf{L}}_Q N \sim R' \otimes^{\mathbf{L}}_Q Q \simeq R' \simeq R' \otimes^{\mathbf{L}}_R R$$

and so (2.9.4) implies $C \sim R$.

(b) Assume that N is dualizing for Q. Because the map $Q \to R'$ is surjective with kernel generated by a Q-regular sequence, we conclude from (2.9.4) that $R' \otimes^{\mathbf{L}}_Q N \simeq R' \otimes^{\mathbf{L}}_R C$ is dualizing for R', and similarly that C is dualizing for R. $\square$

Lemma A.5. *Assume that R is local and let $R \to R' \leftarrow Q$ be a quasi-deformation. Let K and X be R-complexes and let L and Y be Q-complexes such that $R' \otimes^{\mathbf{L}}_Q L \simeq R' \otimes^{\mathbf{L}}_R K$ and $R' \otimes^{\mathbf{L}}_Q Y \simeq R' \otimes^{\mathbf{L}}_R X$. If L is homologically finite over Q and $\mathrm{pd}_Q(L)$ is finite, then*

$$R' \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R C) \simeq \mathbf{R}\mathrm{Hom}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q), N).$$

Proof. Because L and R' are both homologically finite Q-complexes of finite projective dimension, Fact 2.5 implies $\mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_Q L) < \infty$. Also, the fact that $R' \otimes^{\mathbf{L}}_Q L$ is homologically finite over Q implies $\mathrm{pd}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q)) < \infty$ by [8, (2.13)].

Our assumptions yield the second isomorphism in the next sequence while the first and third isomorphisms are standard.

$$\begin{aligned}
R' \otimes^{\mathbf{L}}_R (K \otimes^{\mathbf{L}}_R X) &\simeq (R' \otimes^{\mathbf{L}}_R K) \otimes^{\mathbf{L}}_{R'} (R' \otimes^{\mathbf{L}}_R X) \\
&\simeq (R' \otimes^{\mathbf{L}}_Q L) \otimes^{\mathbf{L}}_{R'} (R' \otimes^{\mathbf{L}}_Q Y) \\
&\simeq (R' \otimes^{\mathbf{L}}_Q L) \otimes^{\mathbf{L}}_Q Y \\
&\simeq \mathbf{R}\mathrm{Hom}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q), Q) \otimes^{\mathbf{L}}_Q Y \\
&\simeq \mathbf{R}\mathrm{Hom}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q), Y)
\end{aligned}$$

The fourth isomorphism is from (2.11.1) and the fifth one is in [14, (1.7.b)]; these rely on the finiteness of $\mathrm{pd}_Q(R' \otimes^{\mathbf{L}}_Q L)$ and $\mathrm{pd}_Q(\mathbf{R}\mathrm{Hom}_Q(R' \otimes^{\mathbf{L}}_Q L, Q))$. $\square$

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